POST MID-SEM

Angular Momentum Conservation

R.T.T. $\left.\frac{dN}{dt}\right|_{sys} = \left.\frac{\partial N}{\partial t}\right|_{cv} + \int_{cs} \rho n (\vec{V}_r \cdot \hat{n})\, dA$

~~$N = \vec{r} \times m\vec{v}$~~ $\quad N = \int \vec{r} \times \vec{v}\, dm$

~~$n = \vec{r} \times \vec{v}$~~ $\quad n = \vec{r} \times \vec{v}$

$$\underbrace{\frac{d}{dt}\int (\vec{r}\times\vec{v})\, dm}_{\Sigma \vec{M}_{sys} \to \Sigma \vec{M}_{cv}} = \frac{\partial}{\partial t}\int_{cv} \rho(\vec{r}\times\vec{v})\, d\forall + \int_{cs} \rho(\vec{r}\times\vec{v})(\vec{V}_r \cdot \hat{n})\, dA$$

For Non-Inertial Reference Frame (xyz): let $N = \int_{sys.} \vec{r} \times \vec{V}_{xyz}\, dm$

$n = \vec{r} \times \vec{V}_{xyz}$.

$$\frac{d}{dt}\int_{sys} (\vec{r}\times\vec{V}_{xyz})\, dm = \frac{\partial}{\partial t}\int_{cv} \rho(\vec{r}\times\vec{V}_{xyz})\, d\forall + \int \rho\, (\vec{r}\times\vec{V}_{xyz})(\vec{V}_r \cdot \hat{n})\, dA$$

For a system of fixed mass that doesn't change with time,

$$LHS = \frac{d}{dt}\int_{sys} (\vec{r}\times\vec{V}_{xyz})\, dm = \int_{sys} \frac{d}{dt}(\vec{r}\times\vec{V}_{xyz})\, dm$$

$$= \int_{sys} \left[\underbrace{\frac{d\vec{r}}{dt}}_{\vec{V}_{xyz}} \times \vec{V}_{xyz} + \vec{r} \times \frac{d}{dt}(\vec{V}_{xyz}) \right] dm$$

($\frac{d\vec{r}}{dt} \times \vec{V}_{xyz}$: null)

$$= \int_{sys} (\vec{r} \times \vec{a}_{xyz})\, dm$$

$$\vec{a}_{xyz} = \vec{a}_{XYZ} - \underbrace{\left[\vec{a}_{cv} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \dot{\vec{\omega}}\times\vec{r} + 2\vec{\omega}\times\vec{V}_{xyz}\right]}_{\vec{a}_{rel}}$$

$$LHS = \int_{sys} \vec{r} \times \vec{a}_{XYZ}\, dm - \int_{sys} (\vec{r}\times\vec{a}_{rel})\, dm$$

$\int_{sys} \vec{r} \times \vec{a}_{XYZ}\, dm = \Sigma \vec{M}_{sys} = \Sigma \vec{M}_{cv}$

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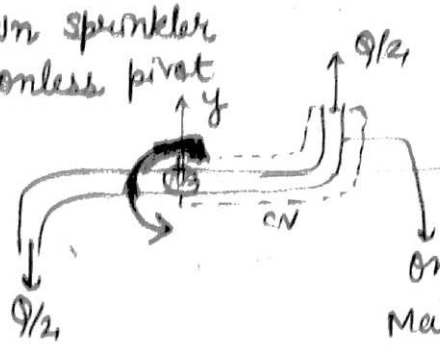
Prob. Lawn sprinkler

- frictionless pivot
- Each arm R radius A cross-section



I very small compared to R

only purpose to divert Making it longer is useless.

$\omega(t) = ?$

Approach 1 : Inertial reference frame

$$\Sigma \vec{M} = \frac{\partial}{\partial t}\int_{CV} \rho(\vec{r}\times\vec{v})\,d\forall + \int_{CS} \rho(\vec{r}\times\vec{v})(\vec{v}_r\cdot\hat{n})\,dA$$

$$\vec{v} = \vec{\omega}\times\vec{r} + \vec{v}_r$$

$$= \omega\hat{k}\times r\hat{i} + v_r\hat{i}$$

$$= \omega r\hat{j} + v_r\hat{i}$$

$$\vec{r}\times\vec{v} = r\hat{i}\times(\omega r\hat{j} + v_r\hat{i})$$

$$= \omega r^2\hat{k}$$

$$\frac{\partial}{\partial t}\int_{sys}\rho(\vec{r}\times\vec{v})\,d\forall = \frac{\partial}{\partial t}\int_{r=0}^{r=R}\rho\omega r^2 A\,dr$$

$$= \frac{\partial}{\partial t}\left(\rho\omega A\frac{R^3}{3}\right)$$

$$= \rho\dot{\omega}A\frac{R^3}{3}$$

$$\int_{CS}\rho(\vec{r}\times\vec{v})(\vec{v}_r\cdot\hat{n})\,dA = \int_{CS}\rho\left[\left(R\hat{i} + \underbrace{\text{small}\,\hat{j}}_{\text{neglect}}\right)\times(\omega R + v_r)\hat{j}\right]$$

$$v_r\hat{j}\,dA$$

$$= \int_{CS}\rho R(\omega R + v_r)\,v_r\,dA\,\hat{k}$$

$$= \rho R(\omega R + v_r)A\,\hat{k}$$

$\left(v_r = \frac{Q}{2A}\right)$

$$\frac{d\omega}{dt} + a\omega = b$$

$$IF = e^{\int a\,dt}$$

$$\therefore\ e^{at}\frac{d\omega}{dt} + a\omega e^{at} = be^{at}$$

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$$\Rightarrow \frac{d}{dt}(\omega e^{at}) = be^{at}$$

$$\Rightarrow \boxed{\omega e^{at} = \frac{be^{at}}{a} + c}$$

Approach 2 : Non Inertial frame; (xyz) attached to sprinkler

$$\cancel{\sum \vec{M}}^{\to 0} - \int_{CV} \rho(\vec{r} \times \vec{a}_{rel})\, d\forall = \frac{\partial}{\partial t}\int_{CV} \rho(\vec{r} \times \vec{V}_{xyz})\, d\forall$$

frictionless

$$+ \int_{CS} \rho(\vec{r} \times \vec{V}_{xyz})(\vec{V}_r \cdot \hat{n})\, dA$$

$$\vec{a}_{rel} = \cancel{\vec{a}_{cv}}^{0} + \vec{\omega} \times (\vec{\omega} \times r) + \dot{\vec{\omega}} \times r + 2\vec{\omega} \times \vec{V}_{xyz}$$

$$= \omega \hat{k} \times (\omega \hat{k} \times r\hat{i}) + \dot{\omega}\hat{k} \times r\hat{i} + 2\omega \hat{k} \times V_{xyz}\hat{i}$$

$$= -\omega^2 r\hat{i} + \dot{\omega} r\hat{j} + 2\omega V_{xyz}\hat{j}$$

$$\vec{r} \times \vec{a}_{rel} = \dot{\omega} r^2 \hat{k} + 2\omega V_{xyz} r \hat{k}$$

$$\int_{CV} \rho(\vec{r} \times \vec{a}_{rel})\, d\forall = \rho\dot{\omega}\int_0^R r^2 A\, dr\, \hat{k} + 2\omega V_{xyz}\int_0^R rA\, dr\, \hat{k}$$

$$= \rho A \dot{\omega}\frac{R^3}{3}\hat{k} + \cancel{2}\omega V_{xyz} A \frac{R^2}{\cancel{2}}\hat{k}$$

$$\int_{CS} \text{——} = \int_{CS} \rho(R\hat{i} \times V_r\hat{j})(V_r\hat{j} \cdot \hat{j})\, dA$$

$$= \rho V_r^2 RA\hat{k}$$

Make a graph of ω vs t

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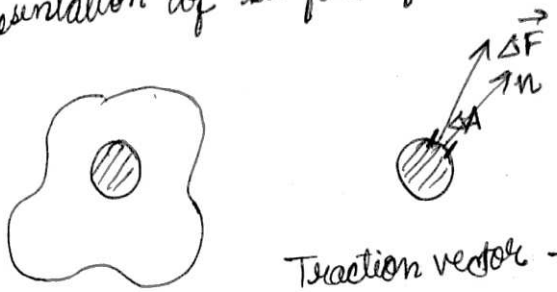
DYNAMICS OF VISCOUS FLOWS

Two types of forces

- surface
- body.

Representation of surface forces.

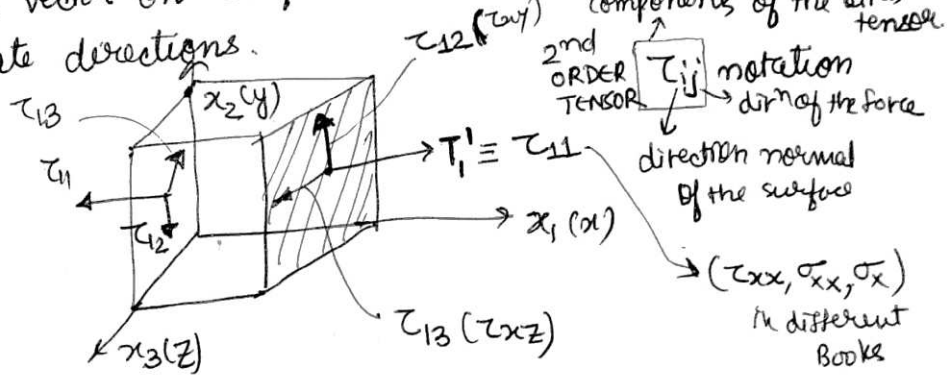


Traction vector $\rightarrow \vec{T}^n = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}}{\Delta A}$

i^{th} component of traction vector.

T_i^n

Traction vector on surfaces with normals along the coordinate directions.

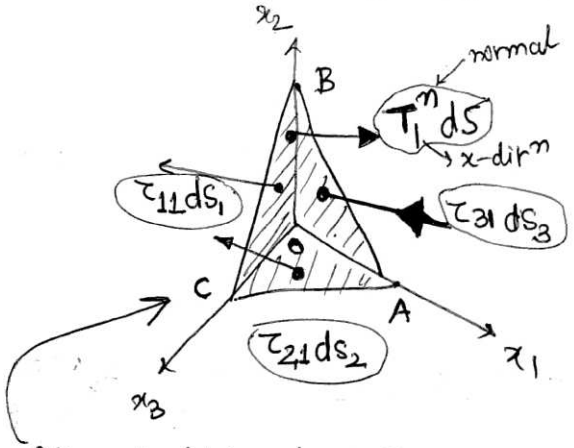


components of the stress tensor

2nd ORDER TENSOR τ_{ij} notation → dirn of the force

direction normal of the surface

SIGN CONVENTION : SHOW FORCES IN +ve direction if the normal itself is in +ve direction.



Represent forces along x_1

OBC → dS_1

OAC → dS_2

OAB → dS_3

ABC → dS

Total Body force along x-dirn

$\Delta V = \frac{1}{3} dS h$

$h \rightarrow \perp$ from O to ABC

$$\Sigma F_1 = \Delta m a_1 \rightarrow -\tau_{11} dS_1 - \tau_{21} dS_2 - \tau_{31} dS_3 + T_1^n dS + b \tfrac{1}{3} dS h$$

$$= \rho \tfrac{1}{3} dS h \, a_1$$

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$n = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$

direction of $ds \rightarrow n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$

" " $ds_1 \rightarrow \hat{i}$

dot product $\rightarrow n_1$

$\therefore \boxed{ds_1 = ds n_1}$ Similarly, $\boxed{ds_2 = ds n_2}$, $\boxed{ds_3 = ds n_3}$

Divide both sides by ds and take lim $h \to 0$

$$T_1^n = \tau_{11} n_1 + \tau_{21} n_2 + \tau_{31} n_3$$

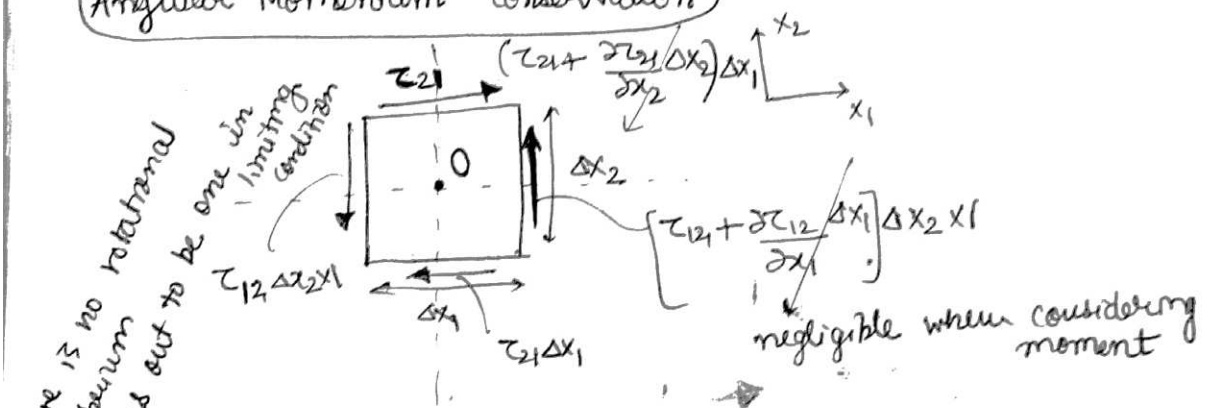
To generalise: $T_i^n = \tau_{1i} n_1 + \tau_{2i} n_2 + \tau_{3i} n_3$

$$= \sum_{j=1}^{3} \tau_{ji} n_j \rightarrow \tau_{ji} n_j \rightarrow \tau_{ij} n_j \quad (\tau_{12} = \tau_{21})$$

$$\begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \end{bmatrix} = \begin{bmatrix} \tau_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

A 2nd order tensor maps one vector onto another.

Angular Momentum Conservation



$$\Sigma M_O = I_O \ddot{\theta}$$

$$\tau_{12}\Delta x_2 \Delta x_1 - \tau_{21}\Delta x_1 \Delta x_2 = \rho \frac{\Delta x_1 \Delta x_2 \times 1}{12}\left[(\Delta x_1)^2 + (\Delta x_2)^2\right]\ddot{\theta}$$

$\Rightarrow$ $\lim_{\Delta x_1, \Delta x_2 \to 0}$ $\boxed{\tau_{12} = \tau_{21}}$

$\therefore$ Only 6 independent tensor to be prescribed.

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Equation of motion

RTT for linear momentum conservation

$$\Sigma \vec{F}_{cv} = \frac{\partial}{\partial t}\int_{CV} \rho \vec{v}\, dV + \int_{CS} \rho \vec{V}\,(\vec{V}_r \cdot \hat{n})\, dA$$

- stationary and non-deformable CV assumed

$$\Sigma F_{cv,i} = \int_{CV} \frac{\partial}{\partial t}(\rho u_i)\, dV + \int_{CS} \rho u_i\,(\vec{v}\cdot\hat{n})\, dA$$

$$\int_{CS} T_i^n\, dA + \int_{CV} b_i\, dV$$

$$T_i^n = \tau_{ij} n_j$$

$$= [\tau_{i1} n_1 + \tau_{i2} n_2 + \tau_{i3} n_3]$$

$$= \underbrace{[\tau_{i1}\hat{i} + \tau_{i2}\hat{j} + \tau_{i3}\hat{k}]}_{\vec{\tau}_i} \cdot [n_1\hat{i} + n_2\hat{j} + n_3\hat{k}]$$

$$= \vec{\tau}_i \cdot \hat{n}$$

$$\int_{CS} (\vec{\tau}_i \cdot \hat{n})\, dA + \int_{CV} b_i\, dV = \int_{CV} \frac{\partial}{\partial t}(\rho u_i)\, dV + \int_{CS} (\rho u_i \vec{V}\cdot\hat{n})\, dA$$

$$\rightarrow \int_{CV} \nabla\cdot\vec{\tau}_i\, dV + \int_{CV} b_i\, dV = \int_{CV} \frac{\partial}{\partial t}(\rho u_i)\, dV + \int_{CV} \nabla\cdot(\rho u_i \vec{V})\, dV$$

$$\nabla\cdot\vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right)$$

$$= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$= \sum_{j=1}^{3} \frac{\partial F_j}{\partial x_i} \rightarrow \frac{\partial F_j}{\partial x_i}$$

$$\rightarrow \int_{CV} \frac{\partial \tau_{ij}}{\partial x_j}\, dV + \int_{CV} b_i\, dV = \int_{CV} \frac{\partial}{\partial t}(\rho u_i)\, dV + \int_{CV} \frac{\partial}{\partial x_j}(\rho u_i u_j)\, dV$$

$$\int_{CV} (\quad \downarrow 0)\, dV = 0$$

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$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial \tau_{ij}}{\partial x_j} + b_i$$ Cauchy eqn / Navier eqn

Unknown ρ, u_1, u_2, u_3 $\tau_{ij} \rightarrow 6$ = 10 unknowns

Independent equation = 3 Momentum + 1 continuity = 4

$$\tau_{ij} = \tau_{ij}^{\text{hydrostatic}} + \tau_{ij}^{\text{Deviatoric}}$$
$$= -p\delta_{ij} + \tau_{ij}^{\text{deviatoric}}$$

$\delta_{ij} \rightarrow$ Cranekar delta

$\delta_{ij} = 1$ if $j = 1$
$= 0$ otherwise

$\tau_{ij}^{\text{deviatoric}} \longrightarrow$ function of $\frac{\partial u_i}{\partial x_j}$

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]}_{e_{ij}} + \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right]}_{\text{rotation}}$$

deformation

$\tau_{ij} \rightarrow$ function of e_{ij}

For Newtonian fluids

τ_{ij} is a linear function of the rate of deformation tensor

$\tau_{ij} = c e_{ij}$ X

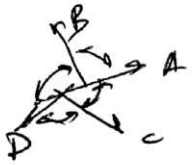
$\tau_{ij} = c_{ijkl} e_{kl}$ ✓ $3^4 =$ 81 material constants required

~~For water, only 2~~

Assume homogeneous, isotropic, Newtonian fluid

Let us convert c_{ijkl} to a homogeneous isotropic scalar

$$S = c_{ijkl} A_i B_j C_k D_l$$
$$= \alpha(\vec{A}\cdot\vec{B})(\vec{C}\cdot\vec{D})$$
$$+ \beta(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})$$
$$+ \gamma(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$$



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$C_{ijkl} A_i B_j C_k D_l = \alpha A_i B_i C_k D_k + \beta A_i C_i B_j D_j + \gamma A_i D_i B_j C_j$

$B_i = B_j \delta_{ij}$ (if $i=j$, δ is 1, else 0)

~~$= \alpha A_i B_j C_k D_l$~~

$= \alpha A_i B_j \delta_{ij} C_k D_l \delta_{kl} + \beta A_i B_j C_k \delta_{ik} D_l \delta_{jl}$

$+ \gamma A_i B_j C_k \delta_{jk} D_l \delta_{il}$

~~$\Rightarrow C_{ijkl} A_i B_j C_k D_l$~~

$$C_{ijkl} = \alpha \delta_{ij}\delta_{kl} + \beta \delta_{ik}\delta_{jl} + \gamma \delta_{jk}\delta_{il}$$

$\tau_{ij}^{\text{deviatoric}} = C_{ijkl}\, e_{kl}$

For homogeneous, isotropic, Newtonian fluids.

$C_{ijkl} = \alpha \delta_{ij}\delta_{kl} + \beta \delta_{ik}\delta_{jl} + \gamma \delta_{jk}\delta_{il}$

$\tau_{ij}^{\text{deviatoric}} = C_{ijkl}\, e_{kl}$ (ij fixed (can play only with k and l))

$= \alpha \delta_{ij} e_{kk} + \beta e_{ij} + \gamma e_{ji}$

$\tau_{ij}^{\text{deviatoric}} = \tau_{ji}^{\text{deviatoric}}$

$\alpha \delta_{ij} e_{kk} + \beta e_{ij} + \gamma e_{ji} = \alpha \delta_{ji} e_{kk} + \beta e_{ji} + \gamma e_{ij}$

$\Rightarrow \boxed{\beta = \gamma}$

$\therefore \tau_{ij}^{\text{deviatoric}} = \alpha \delta_{ij} e_{kk} + \beta (e_{ij} + e_{ji})$

$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad e_{ji} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$

$e_{ij} + e_{ji} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$

$\tau_{ij} = \tau_{ij}^{\text{hydrostatic}} + \tau_{ij}^{\text{deviatoric}}$

$\tau_{ij} = -p\delta_{ij} + \boxed{\alpha} \delta_{ij} e_{kk} + \boxed{\beta}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

β → viscosity of the fluid

α → volumetric dilation coefficient

$e_{kk} = e_{11} + e_{22} + e_{33}$

$= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) + \frac{1}{2}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) + \frac{1}{2}\left(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_3}{\partial x_3}\right)$

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$$= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_k}{\partial x_k}$$

NAVIER EQ.

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x}(\rho u_i u_j) = \frac{\partial \tau_{ij}}{\partial x_j} + b_i$$

For homogeneous, isotropic, Newtonian fluid:

$$\frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[-p\delta_{ij} + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$$

$$= \frac{-\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_j}{\partial x_i}\right)$$

$$= \frac{-\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right) \leftarrow \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)$$

Assuming homogeneous

$$= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_i}\left[(\lambda + \mu)\frac{\partial u_k}{\partial x_k}\right] + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right)$$

$-\frac{2}{3}\mu$ ← 0 for incompressible flow

RELATIONSHIP B/W λ and μ

Let us consider normal stresses

$$\tau_{11} = -p + \lambda \frac{\partial u_k}{\partial x_k} + 2\mu \frac{\partial u_1}{\partial x_1}$$

Viscous normal stress ??

$p \rightarrow$ thermodynamic pressure (obey's TD eqn of state)

$$\tau_{22} = -p + \lambda \frac{\partial u_k}{\partial x_k} + 2\mu \frac{\partial u_2}{\partial x_2}$$

$$\tau_{33} = -p + \lambda \frac{\partial u_k}{\partial x_k} + 2\mu \frac{\partial u_3}{\partial x_3}$$

Mechanical Pressure

$$p_m = -\left(\frac{\tau_{11} + \tau_{22} + \tau_{33}}{3}\right)$$

$$-3p_m = -3p + (3\lambda + 2\mu)\frac{\partial u_k}{\partial x_k}$$

Stoke's Hypothesis assumes mechanical pressure = thermodynamic pressure

$$\therefore \quad 3\lambda + 2\mu = 0 \Rightarrow \boxed{\lambda = -\frac{2}{3}\mu}$$

Different { Stoke's eqn → Nav. Stokes Eqn with inertia terms zero
Stoke's Law → drag force on falling sphere
Stoke's Hypo →

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Putting $\lambda = -\frac{2}{3}\mu$

Eq^n of motion

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(\frac{\mu}{3}\frac{\partial u_k}{\partial x_k}\right)$$

N-S Equation

HOMO
ISO
Newtonian
Stokesian
} fluid

NS Eq^n $\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(\frac{\mu}{3}\frac{\partial u_k}{\partial x_k}\right) + b_i$

Non-linearity

COUPLED SYSTEM OF NON LINEAR PDE

Continuity Eq^n $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$

Unknowns : ρ, u_1, u_2, u_3, p

Equations : continuity + 3 momentum (NS-Eq^n)

Can be closed if ρ is prescribed.

Additional Eq^{n}s

Equation of state $\rho = f(p, T)$

For non-isothermal system, we require additional governing differential equation for temperature [Energy Equation]

In continuity eq^n, we use VOLUME AVG. VELOCITY

NON CONSERVATIVE FORM

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = u_i \frac{\partial \rho}{\partial t} + \rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial}{\partial x_j}(\rho u_j) + \rho u_j \frac{\partial u_i}{\partial x_j}$$

$$= u_i \underbrace{\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right]}_{0 \text{ (Continuity)}} + \rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right]$$

$$= \rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right]$$

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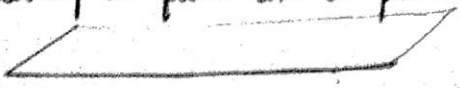
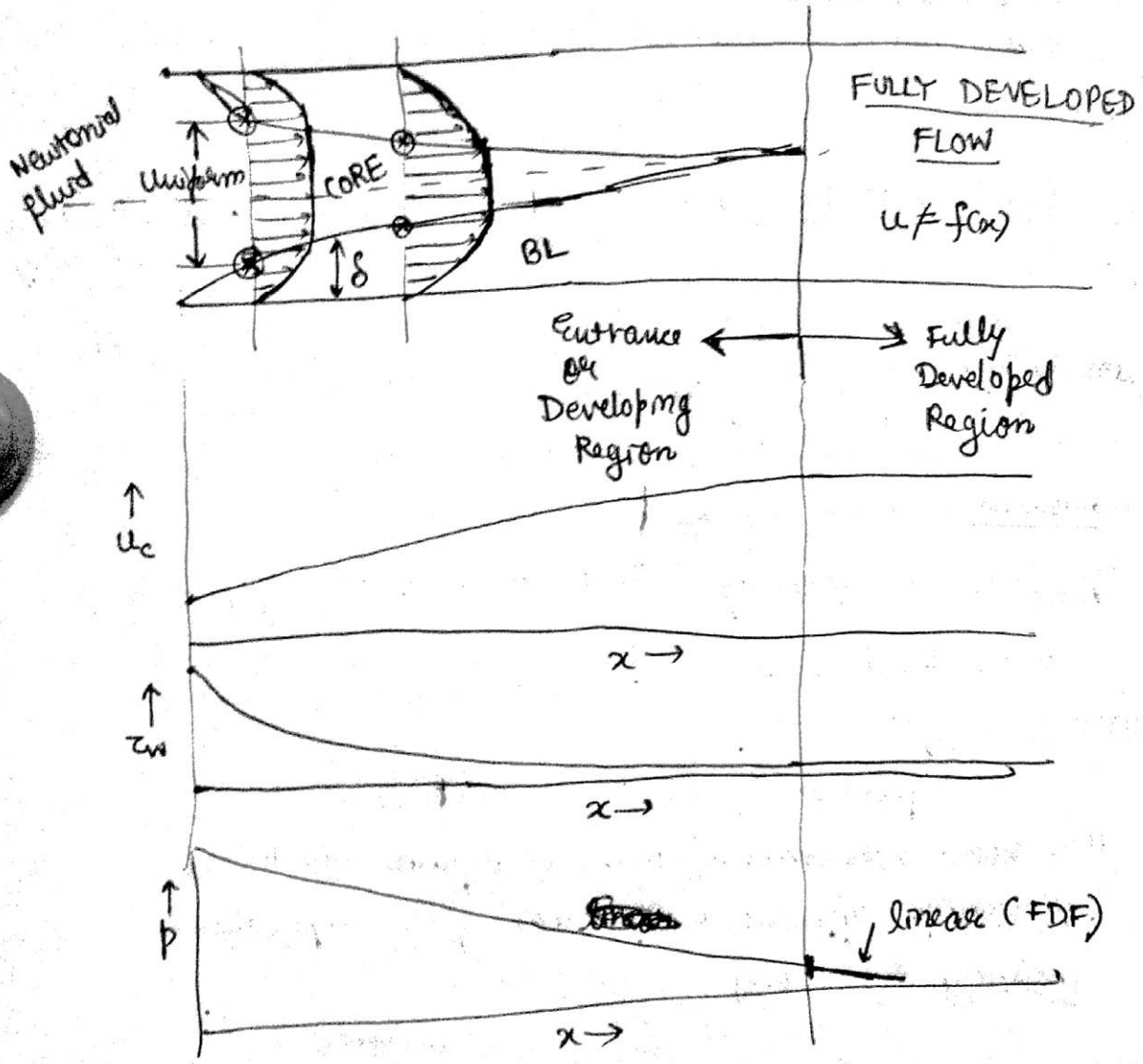
Some application of the Navier stokes equation

Ex1 : Fully developed flow in a parallel plate channel





$\tau_w \sim u \frac{du}{dy}$

$\tau_w \sim \frac{u u_c}{\delta}$ —— increases with x

δ —— increases with x

Just enters the plates τ_w asymptotically very high ($\delta \to 0$)

LHS of NS Eqn $\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right]$

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2D incompressible flow.

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ for FDF

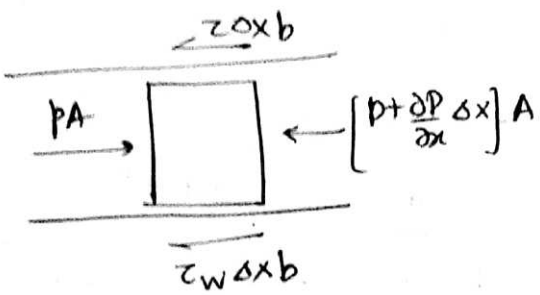
$v \neq f(y)$

$v = 0$ at wall

$\therefore v = 0$ for all y

FDF (Fully Developed flow)

$\therefore \rho\left[\frac{\partial u}{\partial t}^{0(\text{steady})} + u\frac{\partial u}{\partial x}^{0(\text{FDF})} + v\frac{\partial u}{\partial y}_{0}\right]$

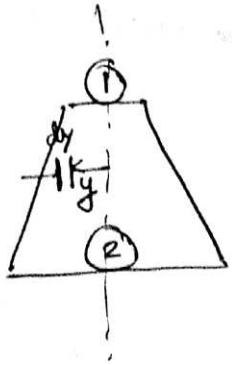


width = b

$\left[p - \left[p + \frac{\partial p}{\partial x}\Delta x\right]\right] A - 2\tau_w \Delta x b = 0$

$\Rightarrow \tau_w = \frac{-A}{2b}\frac{\partial p}{\partial x}$

If $\frac{dp}{dx}$ the sole factor for deciding the direction of flow?



$p_1 < p_2$ Can you tell direction of flow.

Mech. Energy = FE + KE + PE

Energy Decides Flow (High to Low)

$\int \frac{1}{2}\rho u\, dy\, a\, u^2 = KE$

$= \alpha \frac{1}{2}\rho a H \bar{u}^3$

α ← KE correction factor

In industries, most flows are turbulent, so good mixing and hence uniform velocity profile Hence $\alpha \simeq 1$

FE + PE Piezometric Head (due to)

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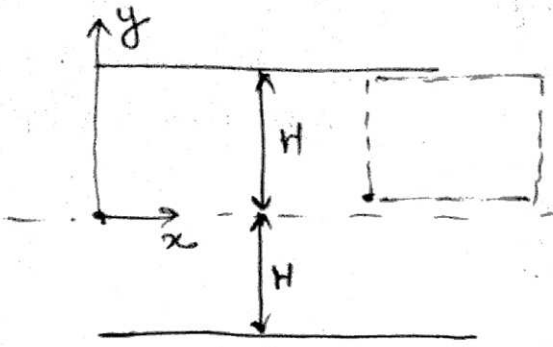
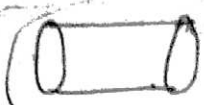
~~FULLY DEVELOPED FLOW~~

Fully Developed velocity profile for a parallel plate channel

(Pressure driven flow)



PLANE POISEULLE FLOW



HAGNEN POISEUBLE FLOW

NS Eqn • Incompressible • steady flow • 2D

No Body force assumed

x - Momentum:

$$\rho\left[\frac{\partial u}{\partial t}^{0} + u\frac{\partial u}{\partial x}^{0(FDF)} + v\frac{\partial u}{\partial y}^{0}\right] = -\frac{\partial p}{\partial x}^{\frac{dp}{dx}} + \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)^{0(FDF)} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \rightarrow \mu\frac{d^2u}{dy^2}$$

y - momentum

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)$$

(0, 0, 0; 0; 0)

$$\Rightarrow \frac{\partial p}{\partial y} = 0$$

$$\Rightarrow p \neq f(y)$$

$$\mu\frac{d^2u}{dy^2} = \frac{dp}{dx}$$

(Viscous effects balance Pressure effects)

function of y only = function of x only

$\Rightarrow$ each = constant = c (say)

$$\therefore \frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}$$

$$\therefore \frac{du}{dy} = cy + c_1$$

$$\Rightarrow \boxed{u = \frac{cy^2}{2} + c_1 y + c_2}$$

B.C.

- $y = 0$, $\frac{du}{dy} = 0$ (due to symmetry) $\Rightarrow c_1 = 0$
- $y = H$, $u = 0$ (No slip) $\therefore c_2 = -\frac{cH^2}{2}$

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$\therefore u = \frac{c}{2}(y^2 - H^2)$ Parabolic in Nature

$$\bar{u} = \frac{\int u\,dA}{\int dA} = \frac{\int_0^H \frac{c}{2}(y^2 - H^2)\,dy\,b}{\int_0^H dy\,b} = \frac{\frac{c}{2}\left(\frac{y^3}{3} - H^2 y\right)\Big|_0^H}{H}$$

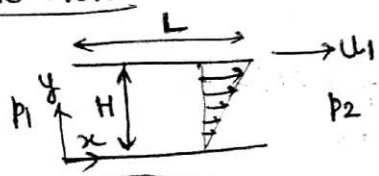
$$= \frac{\frac{c}{2}\left(\frac{H^3}{3} - H^3\right)}{H}$$

$$= -\frac{cH^2}{3}$$

$$\therefore c = -\frac{3\bar{u}}{H^2}$$

$$\therefore \frac{u}{\bar{u}} = \frac{3}{2}\left(1 - \frac{y^2}{H^2}\right)$$

Ex2 Covette Flow



(Shear driven Flow)
(without pressure gradient)
↓
PURE COVETTE FLOW

Similar like Plane Poiseulle Flow

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} = c = \frac{p_2 - p_1}{L}$$

No press. grad: $p_2 = p_1 \therefore \frac{d^2u}{dy^2} = 0 \Rightarrow u = c_1 y + c_2$

$y = 0, u = 0 \Rightarrow c_2 = 0$

$y = H, u = u_1$

$\therefore c_1 = \frac{u_1}{H}$

$$\therefore u = \frac{u_1 y}{H}$$

$$\frac{du}{dy} = \text{constant}$$

COMBINED COVETTE-POISELLE FLOW (press. grad + shear.

$$0 = -\frac{dp}{dx} + \mu\frac{d^2u}{dy^2}$$

$$0 = -\frac{dp}{dx} + \mu\frac{d^2u^{(1)}}{dy^2} \rightarrow \text{Plane Poiseulle}$$

$$0 = -\mu\frac{d^2u^{(2)}}{dy^2} \rightarrow \text{Covette Flow}$$

$$0 = -\frac{dp}{dx} + \mu\frac{d^2}{dy^2}\left[u^{(1)} + u^{(2)}\right]$$

$$\Rightarrow 0 = -\frac{dp}{dx} + \mu\frac{d^2u}{dy^2}$$

JUST ADD THE TWO SOLUTIONS

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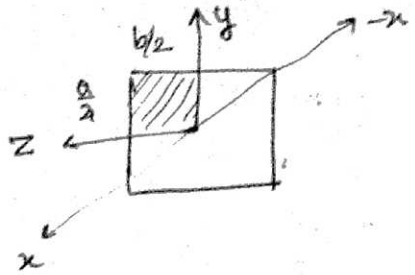
Fully developed pressure-driven flow through rectangular channels.



g-a.e

$$0 = -\frac{dp}{dx} + \mu\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

Method 1: Approximate value

$$u_{app} = c\left(y^2 - \frac{a^2}{4}\right)\left(z^2 - \frac{b^2}{4}\right)$$

$$\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} \mu\left\{\left(\frac{\partial^2 u_{app}}{\partial y^2}\right) + \left(\frac{\partial^2 u_{app}}{\partial z^2}\right)\right\} dy\,dz$$

$$= \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} \frac{dp}{dx}\, dy\,dz$$

$$\frac{\partial u_{app}}{\partial y} = 2cy\left(z^2 - \frac{b^2}{4}\right)$$

$$\Rightarrow \frac{\partial^2 u_{app}}{\partial y^2} = 2c\left(z^2 - \frac{b^2}{4}\right)$$

$$\frac{\partial^2 u_{app}}{\partial y^2} = 2c\left(y^2 - \frac{a^2}{4}\right)$$

$$\mu 2c \int_{-b/2}^{b/2}\left[\int_{-a/2}^{a/2}\left(y^2 - \frac{a^2}{4} + z^2 - \frac{b^2}{4}\right) dy\right] dz = \frac{dp}{dx}\, ab$$

$$\Rightarrow c = ?$$

H.W.: Do it for parallel plate $\left[\frac{a}{b} >> 1 \text{ or } \frac{a}{b} << 1\right]$

Method 2: Exact solution

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\mu}\frac{dp}{dx}$$ Poiseuille eqn

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Let $u = u_1$ be the solution of

$$\frac{d^2u_1}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}$$

$u = u_1 + u_2$ is a soln of the problem

$u = u_2$, be the solution of

$$\frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2} = 0$$

BCs. At $y = 0$, $\frac{du_1}{dy} = 0$

At $y = \frac{a}{2}$, $u_1 = 0$

Soln $\rightarrow u_1(y)$

At $y = 0$ $\quad \frac{\partial u_2}{\partial y} = 0 \quad \left(\frac{\partial u}{\partial y} = 0\right)$ (1)

At $y = \frac{a}{2}$ $\quad u_2 = 0 \quad (u = u_1 + u_2 = 0)$ (2)

At $z = 0$, $\frac{\partial u_2}{\partial z} = 0$ (3)

At $z = \frac{b}{2}$ $\quad u_2 = -u_1(y)$ (4)

Out of 4, 3 BC's are homogeneous.

So Method of seperation of variables can be used

Soln Method of seper. of variables

$$u_2 = f(y)\, g(z)$$

$$-\frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2} = 0$$

$$\Rightarrow f''g + g''f = 0$$

$$\Rightarrow \frac{f''}{f} = -\frac{g''}{g} = -\lambda^2 \text{ (say)}$$

$f'' + \lambda^2 f = 0 \Rightarrow f = C_1 \cos\lambda y + C_2 \sin\lambda y$

$g'' - \lambda^2 g = 0 \Rightarrow g = C_3 \cosh\lambda z + C_4 \sinh\lambda z$

BCs (1) $\rightarrow C_2 = 0$

(3) $\rightarrow C_4 = 0$

$$u_2 = C_1 C_3 \cos\lambda y \cosh\lambda z$$

$$= A \cos\lambda y \cosh\lambda z$$

$$u_2 = \Sigma A_n \cos\lambda_n y \cosh\lambda_n z$$

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(b.c.-2) $\cos \lambda_n \frac{a}{2} = 0$

$\Rightarrow \lambda_n \frac{a}{2} = (2n+1)\frac{\pi}{2}$

$\Rightarrow \lambda_n = \left(\frac{2n+1}{a}\right)\pi$

↳ Eigenvalues → Eigenfunctions

$$\frac{d^2 f_n}{dy^2} + \lambda_n^2 f_n = 0$$

$$\int_0^{a/2} \frac{d^2 f_n}{dy^2} f_m dy + \lambda_n^2 \int_0^{a/2} f_n f_m \, dy = 0$$

$$\left[f_m \frac{df_n}{dy}\right]_0^{a/2} - \int_0^{a/2} \frac{df_m}{dy} \cdot \frac{df_n}{dy} \, dy + \lambda_n^2 \int_0^{a/2} f_n f_m \, dy = 0$$

$\left[f_m \frac{df_n}{dy}\right]_0^{a/2}$ → 0

Swap m & n

$$-\int_0^{a/2} \frac{df_n}{dy} \cdot \frac{df_m}{dy} \, dy + \lambda_m^2 \int_0^{a/2} f_m f_n \, dy = 0$$

Subtract : $(\lambda_n^2 - \lambda_m^2) \int_0^{a/2} f_n f_m \, dy = 0$

ORTHOGONALITY CONDITION $\int_0^{a/2} f_n f_m dy = 0$ if $n \neq m$

$\neq 0$ otherwise

↓

TRIVIAL SOL^N.

BC. : at $z = b/2$, $u_2 = -u_1(y)$

$$-u_1(y) = \Sigma A_n \underbrace{\cos \lambda_n y}_{f_n} \cosh \lambda_n b/2$$

or multiply with f_n and integrate

$$-\int u_1(y) \cos \lambda_m y \, dy = \int_0^{a/2} \Sigma A_n \cos \lambda_n y \cos \lambda_m y \, dy \cosh \lambda_n \frac{b}{2}$$

$$\Rightarrow A_n = \frac{-\int_0^{a/2} u_1(y) \cos \lambda_n y \, dy}{\int_0^{a/2} \cos^2 \lambda_n y \, dy \cosh \lambda_n \frac{b}{2}}$$

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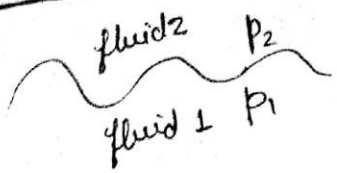
Flows b/w interfaces

What are the BC's at the interface?



KINEMATIC BC

let $f(x,y,z,t)$ represent the interface.

physically, a point on an interface remains on the same in the absense of any cross flow

i.e. $\frac{Df}{Dt} = 0$

$\Rightarrow \frac{\partial f}{\partial t} + \vec{V}_s \cdot \nabla f = 0$

where V_s is the velocity at a point on the interface.

If there is no relative cross velocity between the fluid and the interface, then $\vec{V}_s = \vec{V}$

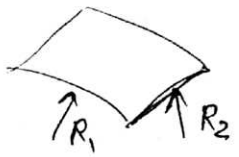
Otherwise there may be a relative velocity

$\vec{V}_r \cdot \hat{n} = \vec{V} \cdot \hat{n} - \vec{V}_s \cdot \hat{n}$ $\qquad \hat{n} = \frac{\nabla f}{|\nabla f|}$

Force balance at the interface:

① Normal force balance

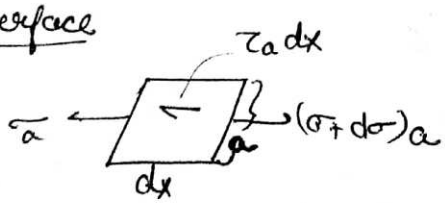
$$\begin{pmatrix} p_1 - p_2 \\ -\vec{T}^n \cdot \hat{n} \text{ deviatoric} \end{pmatrix} = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$



VISCOUS FORCE CAN ALSO GIVE RISE TO NORMAL STRESS

② Tangential force balance.

flat interface



$(\sigma + d\sigma - \sigma)\, a = \tau a\, dx$

$\Rightarrow \boxed{\tau = \frac{d\sigma}{dx}}$ Jump of shear stress across the interface.

$\vec{T}^n \cdot \hat{s} = \boxed{\nabla_s} \sigma \cdot \hat{s}$

↑ surface gradient operator.

$\vec{a}_s = \vec{a} - \vec{a}_n$

$= \vec{a} - (\hat{n} \cdot \vec{a})\hat{n}$

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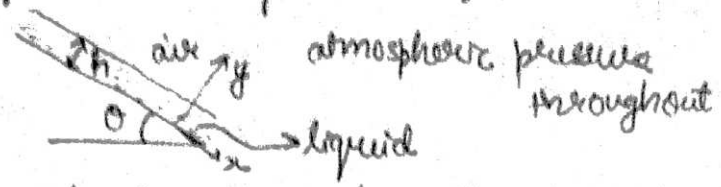
$\nabla_s = \nabla - (\hat{n}\cdot\nabla)\nabla$

Ex: Thin film flow with a flat interface.

So thin that g cosθ component negligible



Assume steady, incompressible, fully developed flow.

x-momentum:

$$0 = \mu\frac{d^2u}{dy^2} - \frac{\partial p}{\partial x} + \rho g\sin\theta$$

$$\mu\frac{d^2u}{dy^2} + \rho g\sin\theta = \frac{\partial p}{\partial x}$$

y-momentum

$$0 = -\frac{\partial p}{\partial y} - \rho g\cos\theta$$

$$\Rightarrow p = -\rho g\cos\theta\, y + f(x)$$

$$\frac{\partial p}{\partial x} = \frac{\partial f}{\partial x}$$

$$\frac{d^2u}{dy^2} = -\frac{1}{\mu}\rho g\sin\theta$$

BCs: at $y=0$, $u=0$ (no slip)

at $y=h$, $\tau=0$

$$\mu_l\frac{du_l}{dy} = \mu_a\frac{du_a}{dy} \quad (\approx 0,\ \mu_a \ll \mu_l)$$

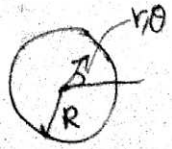
$$\Rightarrow \mu_l\frac{du_l}{dy} = 0$$

FULLY DEVELOPED FLOW THROUGH CIRCULAR PIPE

Continuity: $\frac{\partial\rho}{\partial t}^{0} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r V_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) = 0$

($V_\theta = 0$ by symmetry)

steady, constant density, fully developed flow.



$rV_r \neq$ fn of r

$\Rightarrow rV_r = 0$ Everywhere ($\because rV_r = 0$ at $r=R$)

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r momentum $\quad 0 = -\frac{\partial p}{\partial r}$

θ momentum $\quad 0 = -\frac{1}{r}\frac{\partial p}{\partial \theta}$

z-momentum $\quad \rho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right]$

$= -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$

$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{dv_z}{dr}\right) = \frac{1}{\mu}\frac{dp}{dz} \Rightarrow$ each = constant = c (say)

(fn of r only) (fn of z only)

$r\frac{dv_z}{dr} = \frac{cr^2}{2} + c_1$

bc(1) at $r=0$, $\frac{dv_z}{dr} = 0 \Rightarrow c_1 = 0$

$\frac{dv_z}{dr} = \frac{cr}{2}$

$v_z = \frac{cr^2}{4} + c_1$

bc(2) at wall at $r=R$, $v_z = 0$

$c_1 = -\frac{cR^2}{4}$

$\therefore v_z = \frac{c}{4}(r^2 - R^2)$

$\bar{v}_z = \frac{\int_0^R v_z\, 2\pi r\, dr}{\pi R^2} = \frac{\int_0^R \frac{c}{4}(r^2-R^2)\, 2\pi r\, dr}{\pi R^2} = \frac{c}{2R^2}\left[\frac{R^4}{4} - \frac{R^4}{2}\right]$

$= -\frac{cR^2}{8}$

$\therefore \frac{v_z}{\bar{v}_z} = 2\left(1 - \frac{r^2}{R^2}\right)$ Hagen Poiseuille flow

$\tau_w = -\mu\left.\frac{dv_z}{dr}\right|_{r=R} = \frac{4\mu\bar{v}_z}{R}$

Skin friction coefficient $C_f = \frac{\tau_w}{\frac{1}{2}\rho\bar{v}_z^2} = \frac{16}{\frac{\rho\bar{v}_z D}{\mu}} = \frac{16}{Re_D}$

(C_f: Fanning friction coeff.)

$\bar{v}_z = -\frac{cR^2}{8}$

$c = \frac{1}{\mu}\frac{dp}{dz} = \frac{1}{\mu}\cdot\frac{-h_f\rho g}{L}$ (h_f → head loss due to friction)

$\bar{v}_z = \frac{1}{\mu}\frac{h_f\rho g}{L}\frac{R^2}{8}$

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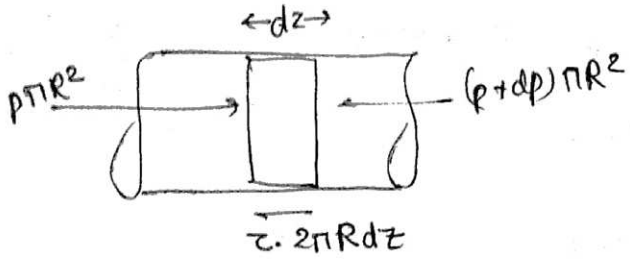
$$h_f = \frac{8\mu \bar{V}_z L}{\rho g R^2} = \frac{8\mu Q L}{\rho g \pi R^4} = \frac{128 \mu Q L}{\rho g \pi D^4}$$ Hagen Poiseuille eqn.

Darcy Friction Factor

$$h_f = f \frac{L}{D} \frac{\bar{V}_z^2}{2g}$$

$$\frac{8\mu \bar{V}_z L}{\rho g R^2} = \frac{fL}{2R} \frac{\bar{V}_z^2}{2g}$$

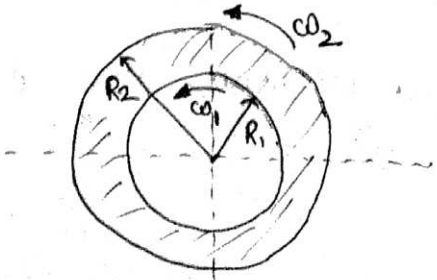
$$\Rightarrow f = \frac{64}{\frac{\rho \bar{V}_z 2R}{\mu}} = \frac{64}{Re_D} = 4C_f$$



$$\tau_w \cdot 2\pi R dz = dp \cdot \pi R^2$$

$$\Rightarrow \tau_w = \frac{R}{2} \cdot \frac{dp}{dz}$$ * only for FDF

Prob Flow b/w two consecutive rotating cylinders (TAYLOR-COUETTE FLOW)



Bearing Lubrication

Assume → long cylinder
→ steady flow
→ Constant Density.

(continuity $\frac{\partial \rho}{\partial t}^{0} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}^{0}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$

$\frac{\partial}{\partial z}(\rho v_z) = 0$ (long cylinder)

$v_r \neq$ not fnc^n of r

r-mom $\rho\left[\frac{\partial v_r}{\partial t}^{0} + v_r \frac{\partial v_r}{\partial r}^{0} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}^{0} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z}^{0}\right] =$ (large length)

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$$-\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\cdot\frac{\partial^2 v_r}{\partial\theta^2} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

$$\therefore \frac{\partial p}{\partial r} = \rho\frac{v_\theta^2}{r}$$

z-mom

$$\rho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta} + v_z\frac{\partial v_z}{\partial z}\right]$$

$$= -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\cdot\frac{\partial^2 v_z}{\partial\theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

$$\Rightarrow \frac{\partial p}{\partial z} = 0$$

θ-mom eqn

$$\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\cdot\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right]$$

$$= -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)\right] + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{z}{r^2}\frac{\partial v_\theta}{\partial z}\Bigg]$$

$$\Rightarrow \frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right) = 0$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}(rv_\theta) = c_1$$

$$\Rightarrow rv_\theta = \frac{c_1 r^2}{2} + c_2$$

$$\Rightarrow v_\theta = c_1\frac{r}{2} + \frac{c_2}{r} = Ar + \frac{B}{r}$$

RANKINE VORTEX (Combo of FREE & Forced)

BC (1) At $r = R_1$; $v_\theta = \omega_1 R_1$
(2) At $r = R_2$; $v_\theta = \omega_2 R_2$
$\Rightarrow A, B.$

$$\boxed{\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]}$$

Ex: Single cylinder rotating in a fluid of infinite extent ($R_2 \to \infty$, $\omega_2 = 0$)

$$v_\theta = \frac{B}{r}, \quad \tau_{r\theta} = \mu r\frac{\partial}{\partial r}\left(\frac{B}{r^2}\right) = \mu r\left(\frac{-2B}{r^3}\right) = -\frac{2\mu B}{r^2}$$

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Shear force per unit volume

$$\mu \nabla^2 \vec{V}$$

Vector Identity

$$\nabla \times (\nabla \times \vec{V}) = \nabla(\vec{\nabla} \cdot \vec{V}) - \nabla^2 \vec{V}$$

($\nabla \times (\nabla \times \vec{V}) = 0$ (incompressible flow); $\vec{\nabla} \cdot \vec{V} = 0$)

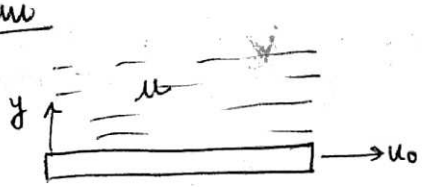
$$0 \Rightarrow \nabla^2 \vec{V} = 0$$

* shear stress not zero but shear force is zero

SOME EXACT SOLUTIONS OF NS EQN for UNSTEADY FLOW.

1. Stokes 1st Problem
2. Stokes 2nd Problem

Stokes 1st Problem



$u(y,t)$

x-momentum

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}^{0} + v\frac{\partial u}{\partial y}^{0} + w^{0}\frac{\partial u}{\partial z}\right]$$

$$= -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2}^{0} + \frac{\partial^2 u}{\partial y^2}\right) + \mu\left(\frac{\partial^2 u}{\partial z^2}\right)$$

$\frac{\partial^2 u}{\partial z^2} \to 0$

$$\Rightarrow \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\eta = \frac{y}{\delta(t)}$$

Assume $\frac{1}{\delta} \sim g(t)$

$$\eta = y\, g(t) \qquad \text{let } \frac{u}{u_0} = f(\eta)$$

$$\frac{\partial u}{\partial t} = u_0 \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = u_0 \frac{df}{d\eta} y\, g'$$

$$\frac{\partial u}{\partial y} = u_0 \frac{df}{d\eta}\frac{\partial \eta}{\partial y} = u_0 \frac{df}{d\eta} g'$$

$$\frac{\partial^2 u}{\partial y^2} = u_0 \frac{d^2 f}{d\eta^2} g^2$$

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g.d.e

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$u_0 \frac{\partial f}{\partial \eta}\left(-\frac{\eta}{2}\right) g' = \nu u_0 \frac{d^2 f}{d\eta^2} g^2$$

$$\frac{\frac{d^2 f}{d\eta^2}}{\eta \frac{df}{d\eta}} = \frac{g'}{\nu g^3} \Rightarrow \text{each} = \text{constant} = c \text{ (say)}$$

f^n of η only; f^n of g only

$$\Rightarrow \frac{dg}{\nu g^3 dt} = c$$

$$\Rightarrow g^{-3} dg = c\nu\, dt$$

$$\Rightarrow \frac{g^{-2}}{-2} = c\nu t + c_1$$

$$\Rightarrow g^2 = \frac{1}{-2c\nu t} \quad [\text{At } t \to 0,\ g \to \infty \Rightarrow c_1 = 0]$$

$c \to -ve$

Ex $\Rightarrow c = -2$

$$g^2 = \frac{1}{4\nu t} \Rightarrow g = \frac{1}{2\sqrt{\nu t}}$$

$$\frac{\frac{d^2 f}{d\eta^2}}{\eta \frac{df}{d\eta}} = -2 \Rightarrow \frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0$$

Let $\frac{df}{d\eta} = h \Rightarrow \frac{dh}{d\eta} + 2\eta h = 0$

$$\Rightarrow \frac{dh}{h} = -2\eta\, d\eta$$

$$\ln h = -\eta^2 + \ln c_2$$

$$\frac{df}{d\eta} = h = c_2 e^{-\eta^2}$$

$$f = c_2 \int_0^\eta e^{-u^2} du + c_3$$

BC's $\eta = 0,\ f = 1$

$\eta \to \infty,\ f = 0$

$c_3 = 1$

$$0 = c_2 \int_0^\infty e^{-u^2} du + 1$$

Let $u^2 = z$

$\Rightarrow u = z^{1/2}$

$\Rightarrow du = \frac{1}{2} z^{1/2 - 1} dz$

$$0 = \frac{1}{2} c_2 \int_0^\infty e^{-z} z^{1/2 - 1} dz + 1$$

$\Gamma(1/2) = \sqrt{\pi}$

$$= c_2 \frac{\sqrt{\pi}}{2} + 1 \Rightarrow c_2 = -\frac{2}{\sqrt{\pi}}$$

$$f = \frac{u}{u_\infty} = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-u^2} du + 1$$

$\text{erf}(\eta)$

$$f = 1 - \text{erf}(\eta) = \text{erfc}(\eta)$$

where $\eta = \frac{y}{2\sqrt{\nu t}}$

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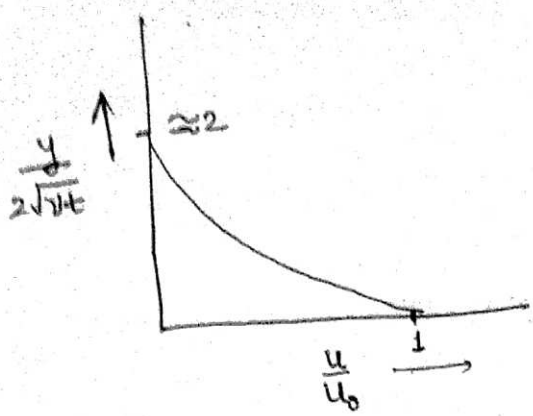
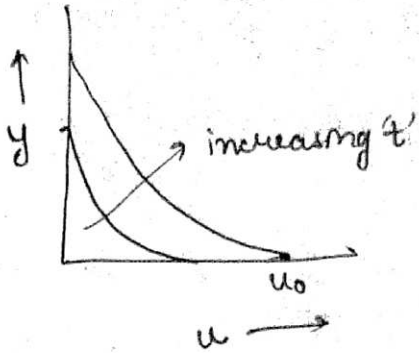
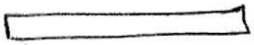




Stokes 2nd Problem

$u = u_0 \sin \omega t$



g.d.e: $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$

$\sim \frac{u_0}{t_s} \quad \sim \frac{\nu u_0}{\delta^2}$

$t_s = \frac{1}{\omega}$

$\delta \sim \sqrt{\nu t_s} \rightarrow \sqrt{\frac{\nu}{\omega}}$

$\bar{t} = t/t_s, \quad \bar{u} = u/u_0, \quad \bar{y} = y/\delta, \quad \delta = \sqrt{\frac{\nu}{\omega}}$

g.d.e. $\frac{\partial u}{\partial \bar{u}} \cdot \frac{\partial \bar{u}}{\partial \bar{t}} \cdot \frac{\partial \bar{t}}{\partial t} = \nu \frac{\partial}{\partial \bar{y}} \left(\frac{\partial u}{\partial \bar{u}} \cdot \frac{\partial \bar{u}}{\partial \bar{y}} \cdot \frac{\partial \bar{y}}{\partial y} \right) \frac{\partial \bar{y}}{\partial y}$

$\frac{u_0}{t_s} \cdot \frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\nu}{\delta^2} u_0 \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$

$\Rightarrow \frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$

Assume $\bar{u} = Im\left[e^{i\bar{t}} f(\bar{y})\right]$

$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$

$i \cdot e^{i\bar{t}} f(\bar{y}) = e^{i\bar{t}} f''(\bar{y})$

$f'' - if = 0$

$f = c_1 e^{\sqrt{i}\bar{y}} + c_2 e^{-\sqrt{i}\bar{y}}$

$e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

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$$\sqrt{i} = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{(1+i)}{\sqrt{2}}$$

$$f = c_1 e^{\frac{1+i}{\sqrt{2}}\bar{y}} + c_2 e^{\frac{-1+i}{\sqrt{2}}\bar{y}}$$

f is finite as $\bar{y} \to \infty$

$(\to 0)$

$\Rightarrow c_1 = 0$

At $\bar{y} = 0$,

$\bar{u} = Im[e^{i\bar{t}}] \to f = 1$

$1 = c_2 e^{\frac{-1+i}{\sqrt{2}}} \to 1$

$\Rightarrow c_2 = 1$

$$f = e^{-\left(\frac{1+i}{\sqrt{2}}\right)\bar{y}}$$

$$\bar{u} = Im\left[e^{i\bar{t}} e^{-\left(\frac{1+i}{\sqrt{2}}\right)\bar{y}}\right]$$

$$= Im\left[e^{i\left[\bar{t} - \frac{\bar{y}}{\sqrt{2}}\right]} e^{-\bar{y}/\sqrt{2}}\right]$$

$$\boxed{\bar{u} = e^{-\bar{y}/\sqrt{2}} \sin\left(\bar{t} - \bar{y}/\sqrt{2}\right)}$$

$\bar{t} = \omega t$

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(3/10/17)

Introduction to turbulence

Reynolds Experiment

↑ dye.

regular, orderly motion → random, dissolved motion

$Re \sim \frac{\text{Inertia force}}{\text{viscous force}} \sim \frac{ma}{\mu \frac{uv}{l} l^2} \sim \frac{\rho l^3 \frac{v^2}{l}}{\mu v l} \rightarrow \frac{\rho v l}{\mu}$ (Reynolds Number)

$ma \sim \rho l^3 \frac{v^2}{l}$, $\frac{uv}{l}$, l^2 → A

Below the CRITICAL Re, flow remains always laminar ✓✓ (Irrespective of the level of disturbance (dampens out)

Beyond the " ", flow may be laminar (if undisturbed) or turbulent (disturbed)

Important char, of turbulent flow

1. random fluctuation of velocity & vorticity
2. Wide range of time scales and length scales.
3. Extreme sensitivity to initial conditions
4. Strongly effective mixing

Energy Cascading

eddy

Largest eddy of system length scale

smallest eddy of diffusive length scale

largest eddy

↑ large eddys pass on the energy to smaller eddy and so on

extracts KE from mean flow and starts rotating

	large eddy	small eddy
length scale	l	η
time scale	u	ν

Rate of extraction of KE

$$\Pi \sim \frac{\frac{1}{2}mu^2}{mt} \sim \frac{u^2}{(t) \sim \frac{l}{u}} \sim \frac{u^3}{l} \quad (1)$$

Rate of dissipation of turbulent KE

$$\epsilon \sim \nu e_{ij} e_{ij} \rightarrow \nu \frac{v^2}{\eta^2} \quad (2)$$

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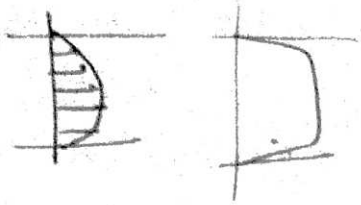
$Re_\eta \sim 1$

$\frac{v\eta}{\nu} \sim 1$ (3)

$v \sim \frac{\nu}{\eta}$



Frnd (2), $\epsilon \sim \frac{\nu v^2}{\eta^2 \eta^2} \Rightarrow \eta \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$

$\pi \sim \epsilon$

$\frac{u^3}{l} \sim \frac{\nu v^2}{\eta^2}$ $\rightarrow \sim \frac{\nu}{\eta}$ from (3)

$\frac{u^3}{l} \sim \frac{\nu^3}{\eta^4} \Rightarrow \eta^4 \sim \left(\frac{u^3}{l\nu^3}\right)^{-1} \Rightarrow \left(\frac{\eta}{l}\right)^4 \sim \left(\frac{ul}{\nu}\right)^{-3} \Rightarrow \frac{\eta}{l} \sim \left(\frac{ul}{\nu}\right)^{-3/4}$

$\Rightarrow \frac{\eta}{l} \sim Re_l^{-3/4}$

Vortex stretching

Vorticity Dynamics

NS eqn

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla p + \mu\nabla^2\vec{v}$$

Vector Identity

$(\vec{v}\cdot\nabla)\vec{v} = \frac{1}{2}\nabla(\vec{v}\cdot\vec{v}) - \vec{v}\times(\nabla\times\vec{v})$

$\vec{v}\times(\nabla\times\vec{v})$ — $\vec{\Omega}$ (vorticity)

$\Rightarrow \nabla\times(\vec{v}\cdot\nabla)\vec{v} = \frac{1}{2}\nabla\times\cancel{(\nabla\vec{v}\cdot\vec{v})}^{0} - \nabla\times(\vec{v}\times\vec{\Omega})$

Vector Identity

$\nabla\times(\vec{v}\times\vec{\Omega}) = \vec{v}(\nabla\cdot\vec{\Omega}) - \vec{\Omega}(\nabla\cdot\vec{v}) + (\vec{\Omega}\cdot\nabla)\vec{v} - (\vec{v}\cdot\nabla)\vec{\Omega}$

$= \vec{v}\cancel{(\nabla\cdot(\nabla\times\vec{v}))}^{0} - \vec{\Omega}\cancel{(\nabla\cdot\vec{v})}^{0} + (\vec{\Omega}\cdot\nabla)\vec{v} - (\vec{v}\cdot\nabla)\vec{\Omega}$

$\nabla\times\nabla p = 0$

$\nabla\times$ both sides of NS eq.

$$\rho\left[\frac{\partial\vec{\Omega}}{\partial t} + (\vec{v}\cdot\nabla)\vec{\Omega} - (\vec{\Omega}\cdot\nabla)\vec{v}\right] = \mu\nabla^2\vec{\Omega}$$

$\Rightarrow \rho\left[\frac{\partial\vec{\Omega}}{\partial t} + (\vec{v}\cdot\nabla)\vec{\Omega}\right] = \mu\nabla^2\vec{\Omega} + \cancel{(\vec{\Omega}\cdot\nabla)\vec{v}}$

large eddy → I (conserved) ⟹ inertia force / viscous force

large eddy = no viscous dissipⁿ

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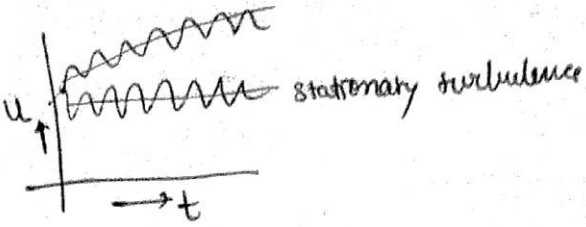
Statistical Representation of turbulent flow

Ex.



Time average of a function f

$$\bar{f} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f\,dt$$

$f = \bar{f} + f'$ Reynolds Decomposition

$\rightarrow \overline{\bar{f}} = \bar{f} + \overline{f'}$

$\Rightarrow \overline{f'} = 0$ (Avg. of fluctuation)

Ex. $\overline{u'} = 0,\ \overline{v'} = 0,\ \overline{w'} = 0,\ \overline{p'} = 0$

But $\overline{u'v'} \neq 0$

o o o o o → u+δu -ve u'
o o o ↓v' → u

But $\overline{uv'} \neq 0$

Space average

$$\bar{f} = \lim_{x\to\infty} \frac{1}{X} \int_0^X f\,dx$$

Ensemble average
Average data over a large no. of repeated experiments.

r.m.s of u' → $\sqrt{\overline{u'^2}}$

Homogeneous turbulence → turbulence statistics are invariant to the ~~translation~~ translation of coordinate axes.

Isotropic turbulence : turbulence statistics are invariant to the rotation, reflection and translation of coordinate axes

Isotropic turbulence, by default, must be homogeneous

Stationary turbulence
→ Ensemble average = time average

Homogeneous turbulence
→ Ensemble average = space average

For homogeneous and stationary turbulence:
time avg = space avg. = ensemble average (Ergodic hypothesis)

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Reynolds Averaging of NS Eqn

DNS
Direct Numerical simulation

let ρ = const.

$$\overline{\frac{\partial}{\partial t}(\rho u_i)} = \overline{\frac{\partial}{\partial t}(\rho(\bar{u}_i + u_i'))}$$

$$= \frac{\partial}{\partial t}(\rho \bar{u}_i)$$

$$\overline{\frac{\partial}{\partial x_j}(\rho u_i u_j)} = \overline{\frac{\partial}{\partial x_j}[\rho(\bar{u}_i + u_i')(\bar{u}_j + u_j')]}$$

$$= \frac{\partial}{\partial x_j}\left[\rho \bar{u}_i \bar{u}_j + \overline{\rho u_i' u_j'}\right.$$

$$\overline{\frac{\partial p}{\partial x_i}} = \overline{\frac{\partial}{\partial x_i}(\bar{p} + p')} = \frac{\partial \bar{p}}{\partial x_i}$$

$$\overline{\frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right)} = \overline{\frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial u_i'}{\partial x_j}\right)\right)}$$

$$= \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{u}_i}{\partial x_j}\right)$$

Doing reynolds averaging of NS Eqn on both sides

$$\frac{\partial}{\partial t}(\rho \bar{u}_i) + \frac{\partial}{\partial x_j}(\rho \bar{u}_i \bar{u}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \bar{u}_i}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(\overline{\rho u_i' u_j'}\right)$$

↓
Reynolds or Turbulence stress

Require closure model for the reynolds stress

$$\overline{\rho u_i' u_j'} = -\mu_T \frac{\partial \bar{u}_i}{\partial x_j}$$

↓
Turbulent viscosity OR Eddy viscosity

Need turbulence models with further eqns to complete this

Ex. $\nu_t = l u u'$

↑
mixing length

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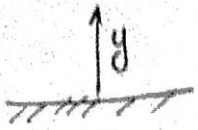
very close to the wall
→ Linear velocity profile



$$\tau_w = \mu \frac{\bar{u}}{y}$$

Define a velocity scale

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

$$u^+ = \frac{\bar{u}}{u_\tau}$$

$$\rho u_\tau^2 = \mu \frac{\bar{u}}{y}$$

$$u^+ = \frac{\bar{u}}{u_\tau} = \frac{\rho u_\tau y}{\mu} = y^+ \text{ (say)} \rightarrow \text{only in the linear region}$$

Inner Layer $\rightarrow u^+ = f(y^+)$

Outer Layer $\rightarrow \frac{u_\infty - u}{u_\tau} = F(y/\delta)$ where δ is the system length scale.

Define $\eta = \frac{y}{\delta}$

In the overlapped region,

$$\left.\frac{d\bar{u}}{dy}\right|_{\text{inner}} = \left.\frac{d\bar{u}}{dy}\right|_{\text{outer}}$$

$$\frac{d\bar{u}}{du^+} \cdot \frac{du^+}{dy^+} \cdot \frac{dy^+}{dy} = \frac{d\bar{u}}{d\eta} \cdot \frac{d\eta}{dy}$$

$\frac{d\bar{u}}{du^+} \rightarrow u_\tau$; $\frac{dy^+}{dy} \rightarrow \frac{\rho u_\tau}{\mu}$; $\frac{d\eta}{dy} \rightarrow 1/\delta$; $\frac{d\bar{u}}{d\eta} \cdot \frac{du^+}{d\eta} \rightarrow u_\tau$

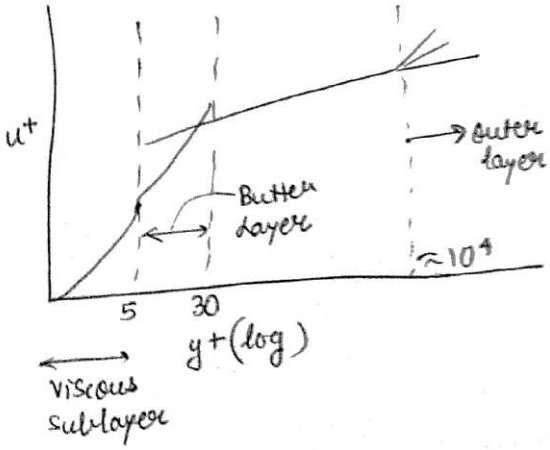
$$\Rightarrow \frac{\rho u_\tau^2}{\mu} \cdot \frac{du^+}{dy^+} = \frac{u_\tau}{\delta} \frac{du^+}{d\eta}$$

$$\Rightarrow \frac{\rho u_\tau^2}{\mu} \frac{\rho}{} \frac{du^+}{dy^+} = \frac{u_\tau y}{\delta} \cdot \frac{du^+}{d\eta}$$

$$\Rightarrow \underbrace{y^+ \frac{du^+}{dy^+}}_{\text{fn of } y^+ \text{ only}} = \underbrace{\eta \frac{du^+}{d\eta}}_{\text{fn of } \eta \text{ only}} \Rightarrow \text{each} = \text{constant} = \frac{1}{K}$$

$$u^+ = \frac{1}{K} \ln y^+ + A \rightarrow \text{log law}$$

$K \approx 0.4$



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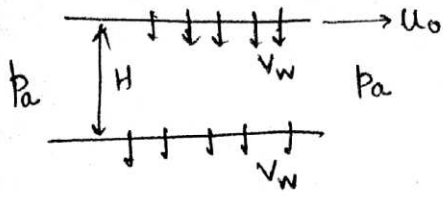
6/11/12 An intro. to the method of perturbation

Prob



2D - incompressible, steady flow, Fully developed flow

Continuity.

$$\underbrace{\frac{\partial u}{\partial x}}_{0} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow v \neq v(y)$$

$$\Rightarrow v = -V_w \text{ at } y = 0, H$$

$$\Rightarrow -v = -V_w \text{ everywhere}$$

y-mom: $\frac{\partial p}{\partial y} = 0 \Rightarrow p \neq p(y)$

x-mom

$$\rho\left[\underbrace{\frac{\partial u}{\partial t}}_{0} + u\underbrace{\frac{\partial u}{\partial x}}_{0} + \underbrace{(v)}_{-V_w}\frac{\partial u}{\partial y}\right] = -\underbrace{\frac{\partial p}{\partial x}}_{\frac{dp}{dx}} + \mu\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{0} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\underbrace{\mu\frac{d^2u}{dy^2} + \rho V_w\frac{du}{dy}}_{\text{fn. of } y \text{ only}} = \underbrace{\frac{dp}{dx}}_{x\text{-only}}$$

each = constant = $\frac{p_u - p_u}{L} = 0$

$\bar{u} = \frac{u}{u_0}$, $\bar{y} = \frac{y}{H}$

$$\mu\frac{u_0}{H^2}\frac{d^2\bar{u}}{d\bar{y}^2} + \rho\frac{V_w u_0}{H}\frac{d\bar{u}}{d\bar{y}} = 0$$

$$\frac{d^2\bar{u}}{d\bar{y}^2} + \left(\frac{\rho V_w H}{\mu}\right)\frac{d\bar{u}}{d\bar{y}} = 0$$

$\frac{\rho V_w H}{\mu} \to t$ (say)

BCs

① At $\bar{y} = 0$, $\bar{u} = 0$

② At $\bar{y} = 1$, $\bar{u} = 1$

Discard the overbars and write

$$\frac{d^2u}{dy^2} + t\frac{du}{dy} = 0$$

At $y = 0$, $u = 0$

$y = 1$, $u = 1$

Case I: t is small ($\to 0$)

Expand: $u = u_0 + tu_1 + t^2u_2 + \cdots$

$$0 = \frac{d^2}{dy^2}(u_0 + tu_1 + t^2u_2 + \cdots) + t\left(\frac{du_0}{dy} + t\frac{du_1}{dy} + t^2\frac{du_2}{dy}\right.$$

Equate terms with like powers of t.

$$\frac{d^2u_1}{dy^2} + \frac{du_0}{dy} = 0 \quad [O(t)] \qquad \frac{d^2u_0}{dy^2} = 0 \quad [O(1)]$$

$\frac{du_0}{dy} = c_1 \Rightarrow u_0 = c_1y + c_2$

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At $y=0$, $u_0=0 \Rightarrow c_2=0$

At $y=1$, $u_0=1 \Rightarrow c_1=y$

$\Rightarrow u_0 = y$

$$\frac{d^2u_1}{dy^2} + \frac{du_0}{dy}^{1} = 0 \quad \Rightarrow \frac{d^2u_1}{dy^2} = -1$$

$$\therefore \frac{du_1}{dy} = -y + c_3 \Rightarrow u_1 = -\frac{y^2}{2} + c_3 y + c_4$$

BC's At $y=0$, $u_1=0 \longrightarrow c_4=0$

At $y=1$, $u_1=0 \Rightarrow 0 = -\frac{1}{2} + c_3 \Rightarrow c_3 = \frac{1}{2}$

$$u_1 = \frac{1}{2} y(1-y)$$

$$u = y + \frac{t}{2} y(1-y) + \cdots$$

How does it compare with exact solution?

$$\frac{d^2u}{dy^2} + t\frac{du}{dy} = 0 \qquad u = e^{my} \text{ trial soln}$$

$$m^2 + tm = 0$$

$$\Rightarrow m(m+t) = 0 \Rightarrow m = 0, -t$$

$$\therefore u = A + B^{-ty}$$

BC's ① At $y=0$, $u=0$

② At $y=1$, $u=1$

$$0 = A + B \Rightarrow B = -A$$

$$1 = A + Be^{-t}$$

$$\Rightarrow A(1-e^{-t}) = 1 \quad \Rightarrow A = \frac{1}{1-e^{-t}}$$

Exact soln $\quad u = \dfrac{1-e^{-ty}}{1-e^{-t}}$

Define $-t = k \quad \therefore u = \dfrac{e^{ky}-1}{e^{k}-1}$

$$\lim_{k\to 0} \frac{e^{ky}-1}{ky} \times \lim_{k\to 0} \frac{k}{e^{k}-1} \times y = y$$

Case 2 $\quad t \to \infty$

$$\frac{d^2u}{dy^2} + t\frac{du}{dy} = 0$$

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Let $\delta = \frac{1}{t}$

$$\frac{du}{dy} + \delta \frac{d^2u}{dy^2} = 0$$

Let $u = u_0 + \delta u_1 + \cdots$

$$\frac{du_0}{dy} + \delta \frac{du_1}{dy} + \delta \frac{d^2u_0}{dy^2} + \delta^2 \frac{d^2u_1}{dy^2} + \cdots = 0$$

$O(1) \Rightarrow \frac{du_0}{dy} = 0 \Rightarrow u_0 = c_5$

BC(s) : At $y = 0, u_0 = 0$
$y = 1, u_0 = 1$ }] X

Rescale : $\eta = y/\delta$ → stretching transformation

$$\frac{du}{dy} \cdot \frac{dy}{d\eta}^{1/\delta} + \frac{\delta}{\delta^2} \frac{d^2u}{d\eta^2} = 0$$

$$\frac{du}{d\eta} + \frac{d^2u}{d\eta^2} = 0$$

$u = e^{\alpha \eta}$ trial soln

$\alpha + \alpha^2 = 0$

$\alpha = 0, -1$

$u = C + De^{-\eta}$

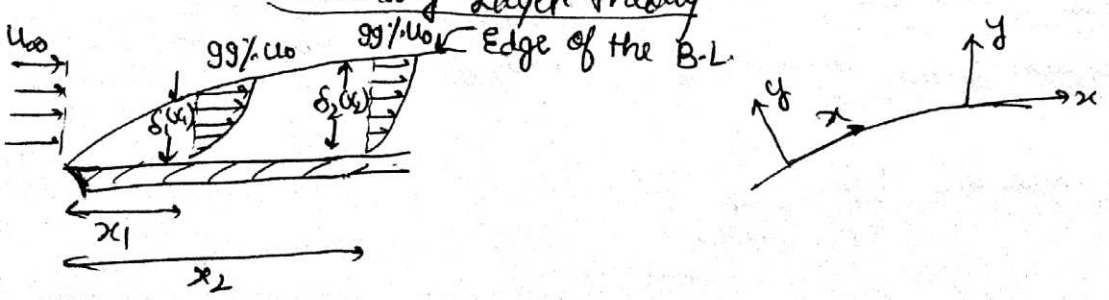
BC's (1) At $\eta = 0, u = 0$

$D = -C$

$u = C(1 - e^{-\eta})$

At $\eta \to \infty, u \to 1 \Rightarrow C = 1$

Boundary Layer theory



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Steady 2-D incompressible flow

continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Order of Magnitude analysis $\sim$ $\frac{u_\infty}{h}$ $\frac{v_\delta}{\delta}$

$$v_\delta \sim \frac{u_\infty \delta}{h}$$

$v_\delta \ll u_\infty$ if $\delta \ll L$

x-mom.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$\frac{\partial p}{\partial x} = \frac{dp_\infty}{dx}$

$\sim \frac{u_\infty^2}{h}$ $\sim \frac{v_\delta u_\infty}{\delta}$ (u_∞/L) $\sim \frac{u_\infty^2}{h}$ $\sim \frac{1}{\rho}\frac{\Delta p_x}{L}$ $\sim \frac{\nu u_\infty}{h^2}$ $\sim \frac{\nu u_\infty}{\delta^2}$

y-mom

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$\sim \frac{u_\infty v_\delta}{h}$ $\sim \frac{v_\delta^2}{\delta}$ $\sim \frac{u_\infty v_\delta}{h}$ $\frac{1}{\rho}\frac{\Delta p_y}{\delta}$ $\sim \frac{\nu v_\delta}{h^2}$ $\sim \frac{\nu v_\delta}{\delta^2}$

Neglect if $\delta \ll L$

$$\frac{\text{Inertia terms in y-mom}}{\text{Inertia terms in x-mom}} \sim \frac{\frac{u_\infty v_\delta}{h}}{\frac{u_\infty^2}{h}} \sim \frac{v_\delta}{u_\infty} \sim \frac{\delta}{h}$$

$$\frac{\text{Viscous terms in y-mom}}{\text{Viscous terms in x-mom}} \sim \frac{\nu v_\delta/\delta^2}{\nu u_\infty/\delta^2} \sim \frac{v_\delta}{u_\infty} \sim \frac{\delta}{L}$$

If $\delta \ll h$, $\frac{\partial p}{\partial y} \ll \frac{\partial p}{\partial x}$

$\simeq 0$

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Boundary layer Equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

Assumptions.

prev. ones + 1) $\delta << L$

2) There is no B.L. separation

Ex: Flow over a flat plate

$$p_\infty + \frac{1}{2}\rho u_\infty^2 = \text{constant} \quad ?$$

$$\frac{dp_\infty}{dx} + \rho u_\infty \boxed{\frac{du_\infty}{dx}}^{\to 0} = 0 \qquad [\text{For flow over a flat plate}]$$

Inertia ~ Viscous

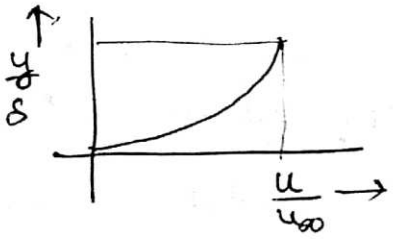
$$\frac{u_\infty^2}{h} \sim \frac{\nu u_\infty}{\delta^2}$$

$$\frac{\delta^2}{h^2} \sim \frac{\nu}{u_\infty L} \Rightarrow \frac{\delta}{L} \sim Re_L^{-1/2}$$

Similarity solution for BL analysis over a flat plate

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$



$\frac{u}{u_\infty}$ is a single valued function of $\frac{y}{\delta}$

$$\frac{y}{\delta} \to \eta$$

$$\frac{u}{u_\infty} = f(\eta) \quad \text{where} \quad \eta = y\,g(x)$$

$$\frac{\partial u}{\partial x} = u_\infty \frac{df}{d\eta}\cdot\frac{\partial \eta}{\partial x}$$

$$= u_\infty \frac{df}{d\eta}\cdot y g'$$

$$\frac{\partial u}{\partial y} = u_\infty \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\infty \frac{df}{d\eta} g$$

$$\frac{\partial^2 u}{\partial y^2} = u_\infty \frac{d^2 f}{d\eta^2} g^2$$

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$$v = \frac{\nu \frac{\partial^2 u}{\partial y^2} - u\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$$

$$\Rightarrow v = \frac{\nu u_\infty \frac{d^2 f}{d\eta^2} g^2 - u_\infty^2 f \frac{df}{d\eta} y g'}{u_\infty \frac{df}{d\eta} g}$$

$$\Rightarrow v = \frac{\nu \frac{d^2 f}{d\eta^2} g}{\frac{df}{d\eta}} - u_\infty f y \frac{g'}{g}$$

$$\frac{\partial v}{\partial y} = \nu \frac{d}{d\eta}\left[\frac{\frac{d^2 f}{d\eta^2}}{\frac{df}{d\eta}}\right] \frac{\partial \eta}{\partial y} g - u_\infty \frac{df}{d\eta} \cdot \frac{\partial \eta}{\partial y} y \frac{g'}{g} - u_\infty f g'/g$$

$$= -\frac{\partial u}{\partial x} \text{ (from continuity)}$$

$$= -u_\infty \frac{df}{d\eta} y g'$$

$$\nu \frac{d}{d\eta}\left[\frac{\frac{d^2 f}{d\eta^2}}{df/d\eta}\right] g^2 = u_\infty f g'/g$$

$$\Rightarrow \frac{1}{f}\frac{d}{d\eta}\left[\frac{\frac{d^2 f}{d\eta^2}}{\frac{df}{d\eta}}\right] = \underbrace{\frac{u}{\nu} g^{-3} \frac{dg}{dx}}_{\text{fn of } x \text{ only}}$$

fn of η only $\Rightarrow$ each = constant = c (say)

$$\frac{u_\infty}{\nu} g^{-3} dg = c\,dx$$

Integrate $\frac{u_\infty}{\nu} \frac{g^{-2}}{-2} = cx + 1$

As $x \to 0$, $g \to \infty \Rightarrow c_1 = 0$

$$g = \sqrt{\frac{u_\infty}{-2c\nu x}} \sim \frac{1}{\delta}$$

c must be $-ve$

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Eg $c = -1/2$ $\quad \therefore g = \sqrt{\frac{u_\infty}{\nu x}}$

$$\frac{1}{f}\frac{d}{dx}\left[\frac{\frac{d^2f}{d\eta^2}}{\frac{df}{d\eta}}\right] = -1/2$$

$$\frac{d}{d\eta}\left[\frac{\frac{d^2f}{d\eta^2}}{\frac{df}{d\eta}}\right] + \frac{1}{2}f = 0$$

Let $dF = f\,d\eta$

$$\frac{d}{d\eta}\left[\frac{F'''}{F''}\right] + \frac{1}{2}dF = 0 \qquad \frac{u}{u_\infty} = \frac{dF}{d\eta} \quad (f)$$

$$F''' + \frac{1}{2}FF'' = k$$

At $\eta = 0$, F = constant (say 0 ; NOT A MUST)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

u: 0 at $y=0$; v: 0 at $y=0$; $\frac{\partial^2 u}{\partial y^2}$: 0 at $y=0$

$\Rightarrow k = 0$

$$\boxed{F''' + \frac{1}{2}FF'' = 0} \Rightarrow \text{Blasius eq}^n$$

BCs

1. At $\eta = 0$, $F = 0$
2. At $\eta = 0$, $F' = 0$ (no slip)
3. At $\eta \to \infty$, $F' = 1$

$y = x^2$

Shooting method
RK-4 ode 2.3
ode 4,5

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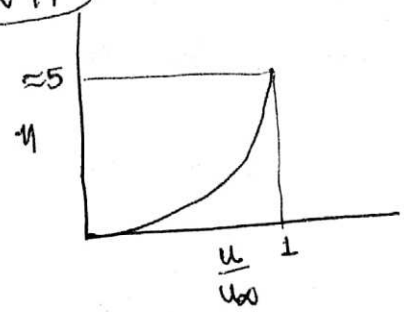
7 Nov 17



$$\frac{u}{u_\infty} = \frac{dF}{d\eta}$$

$\frac{u}{u_\infty} = 1$ at $\eta \approx 5$

$\eta = y\, g(x)$

$g(x) = \sqrt{\frac{u_\infty}{\nu x}}$

$\therefore \quad 5 = \delta\sqrt{\frac{u_\infty}{\nu x}}$

$\Rightarrow \frac{\delta}{x} = 5\sqrt{\frac{\nu x}{u_\infty x^2}}$

$\Rightarrow \boxed{\frac{\delta}{x} = 5\,Re_x^{-1/2}}$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$= \mu \left.\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right|_{\eta=0}$$

$$= \mu u_\infty F''(0)\sqrt{\frac{u_\infty}{\nu x}}$$

Skin friction coefficient:

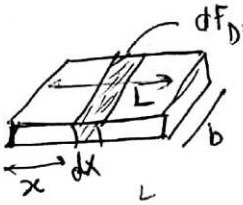
$$C_{fx} = \frac{\tau_w}{\frac{1}{2}\rho u_\infty^2} = \frac{\mu u_\infty F''(0)\sqrt{\frac{u_\infty}{\nu x}}}{\frac{1}{2}\rho u_\infty^2}$$

$$C_{fx} = 2F''(0)\sqrt{\frac{u_\infty \nu^2}{\nu x\, u_\infty^2}}$$

$$= 2\underbrace{F''(0)}_{0.332} Re_x^{-1/2}$$

$$\boxed{C_{fx} = 0.664\, Re_x^{-1/2}}$$

Total drag force on the plate



$$F_D = \int_0^L \tau_w b\, dx$$

$$= \int_0^L \mu u_\infty F''(0)\sqrt{\frac{u_\infty}{\nu x}}\, b\, dx$$

$$= \mu u_\infty \sqrt{\frac{u_\infty}{\nu}}\, F''(0)\, b \int_0^L x^{-1/2}\, dx$$

$$= \mu u_\infty \sqrt{\frac{u_\infty}{\nu}}\, b F''(0)\, 2L^{1/2}$$

∴ skin friction drag coeff

$$C_D = \frac{F_D}{\left(\frac{1}{2}\rho u_\infty^2\right) bL}$$

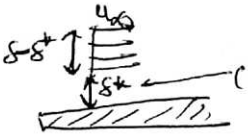
$$= \frac{\mu u_\infty \sqrt{\frac{u_\infty}{\nu}}\, b F''(0)\, 2L^{1/2}}{\frac{1}{2}\rho u_\infty^2\, bL}$$

$$C_D = 4F''(0)\sqrt{\frac{\nu^2 u_\infty}{u_\infty^2 L}}$$

$$= 4F''(0)\, Re^{-1/2}$$

$$\boxed{C_D = 1.328\, Re_L^{-1/2}}$$

DISPLACEMENT THICKNESS



(hypothetical displacement of the boundary)

$$\int_0^\delta u\, dy = (\delta - \delta^*) u_\infty$$

$$\delta^* = \delta - \int_0^\delta \frac{u}{u_\infty} dy$$

$$\delta^* = \int_0^\delta dy - \int_0^\delta \frac{u}{u_\infty} dy$$

$$\boxed{\delta^* = \int_0^\delta \left(1 - \frac{u}{u_\infty}\right) dy}$$

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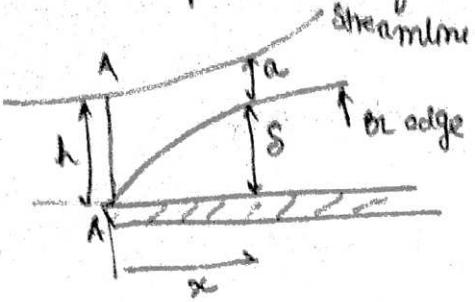
Other interpretations of δ^*



$$hu_\infty = \int_0^\delta u\,dy + a u_\infty$$

displacement of the streamline

$$= \delta + a - h$$

$$(a-h)u_\infty = -\int_\delta^0 u\,dy$$

$$\Rightarrow a - h = \int_0^\delta \frac{u}{u_\infty}\,dy$$

$$= \delta - \int_0^\delta \frac{u}{u_\infty}\,dy$$

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{u_\infty}\right)dy$$

Momentum thickness

Momentum flux across AA:
(per unit mass)

$$\rho u_\infty^2 h$$

Momentum flux across BB:
(per unit width)

$$\int_0^\delta \rho u^2\,dy + \rho u_\infty^2 a$$

$$\rho u_\infty^2 \theta = \rho u_\infty^2 h - \left[\rho \int_0^\delta u^2\,dy + \rho u_\infty^2 a\right]$$

$$= \rho u_\infty^2 \int_0^\delta \frac{u}{u_\infty}\,dy - \rho \int_0^\delta u^2\,dy$$

$\rightarrow h - a$

$$\Rightarrow \theta = \int_0^\delta \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right)dy$$

VON KARMAN Momentum Integral Method

Eg. flow over a flat plate

x-mom (BL eq)

$$\int_0^\delta u\frac{\partial u}{\partial x}\,dy + \int_0^\delta v\frac{\partial u}{\partial y}\,dy = \int_0^\delta \nu\frac{\partial^2 u}{\partial y^2}\,dy$$

term 1 + term 2 = term 3

term 2 : By Parts

$$\int_0^\delta v\frac{\partial u}{\partial y}\,dy$$

$$= [vu]_0^\delta - \int_0^\delta \frac{\partial v}{\partial y}\,u\,dy \qquad \left(-\frac{\partial u}{\partial x}\ \text{(2D+ incomp}\right.$$

$$= v_\delta u_\infty + \int_0^\delta u\frac{\partial u}{\partial x}\,dy$$

$$LHS = \text{term 1} + \text{term 2}$$

$$= \int_0^\delta 2u\frac{\partial u}{\partial x}\,dy + v_\delta u_\infty$$

$$= \int_0^\delta \frac{\partial}{\partial x}(u^2)\,dy + v_\delta u_\infty$$

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Continuity eqn

$$\int_0^\delta \frac{\partial u}{\partial x}dy + \int_0^\delta \frac{\partial v}{\partial y}dy = 0$$

$$\Rightarrow \int_0^\delta \frac{\partial u}{\partial x}dy + v_\delta = 0$$

$$\Rightarrow v_\delta = -\int_0^\delta \frac{\partial u}{\partial x}dy$$

Ex $F = u^2 - uu_\infty$

$a = 0$

$b = \delta$

$$F(x,b) = u_\infty^2 - u_\infty u_\infty = 0$$

$$= \frac{d}{dx}\int_0^\delta (u^2 - uu_\infty)dy$$

$$\text{RHS} = \nu\int_0^\delta \frac{\partial^2 u}{\partial y^2}dy = \left[\nu\frac{\partial u}{\partial y}\right]_\delta - \left[\nu\frac{\partial u}{\partial y}\right]_0$$

$$= 0 - \nu\frac{\partial u}{\partial y}\Big|_0$$

LHS = RHS

$$\Rightarrow \frac{d}{dx}\int_0^\delta (uu_\infty - u^2)dy = \nu\frac{\partial u}{\partial y}\Big|_0 = \frac{\tau_w}{\rho}$$

$$\boxed{\frac{d}{dx}\int_0^\delta \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right)dy = \frac{\tau_w}{\rho u_\infty^2}}$$

VON KARMAN'S MOMENTUM INTEGRAL EQUATION.

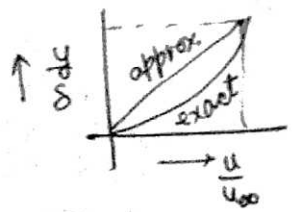
$$\text{LHS} = \int_0^\delta \frac{\partial}{\partial x}(u^2)dy - \int_0^\delta u_\infty \frac{\partial u}{\partial x}dy$$

$$= \int_0^\delta \frac{\partial}{\partial x}(u^2 - uu_\infty)dy$$

Leibnitz (Differentiation under integral sign)

$$\frac{d}{dx}\int_{a(x)}^{b(x)} F(x,y)dy$$

$$= \int_{a(x)}^{b(x)} \frac{\partial F}{\partial x}dy + F(x,b)\frac{db}{dx} - F(x,a)\frac{da}{dx}.$$



Further calculation depends on the choice of velocity profile

Ex $\frac{u}{u_\infty} = a_0 + a_1\left(\frac{y}{\delta}\right) + a_2\left(\frac{y}{\delta}\right)^2 + a_3\left(\frac{y}{\delta}\right)^3$

BC ① At $y=0, u=0$

② At $y=\delta, u=u_\infty$

③ At $y=\delta, \frac{\partial u}{\partial y}=0$

④ At $y=\delta, \frac{\partial^2 u}{\partial y^2}=0$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$0 = a_0$ $\quad a_0 = 0$

$1 = a_0 + a_1 + a_2 + a_3$

$0 = \frac{a_1}{\delta} + \frac{2a_2}{\delta} + \frac{3a_3}{\delta}$

$0 = a_2$

$$\therefore \frac{u}{u_0} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

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$$\frac{d}{dx}\int_0^{\delta}\left[\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[1-\frac{3}{2}\left(\frac{y}{\delta}\right)+\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]dy=\frac{\mu\frac{3}{2}\frac{u_\infty}{\delta}}{\rho u_\infty^2}$$

Assume $\frac{y}{\delta}=\eta$

$$\frac{d\delta}{dx}\int_0^1\left(\frac{3}{2}\eta-\frac{1}{2}\eta^3\right)\left(1-\frac{3}{2}\eta+\frac{1}{2}\eta^3\right)d\eta=\frac{\frac{3}{2}\frac{\mu}{\delta}u_\infty}{\rho u_\infty^2}$$

$\downarrow$

A (say)

$$A\frac{d\delta}{dx}=\frac{3}{2}\frac{\mu}{\rho u_\infty \delta}$$

$$\Rightarrow \delta\, d\delta=\frac{3\nu}{2Au_\infty}dx$$

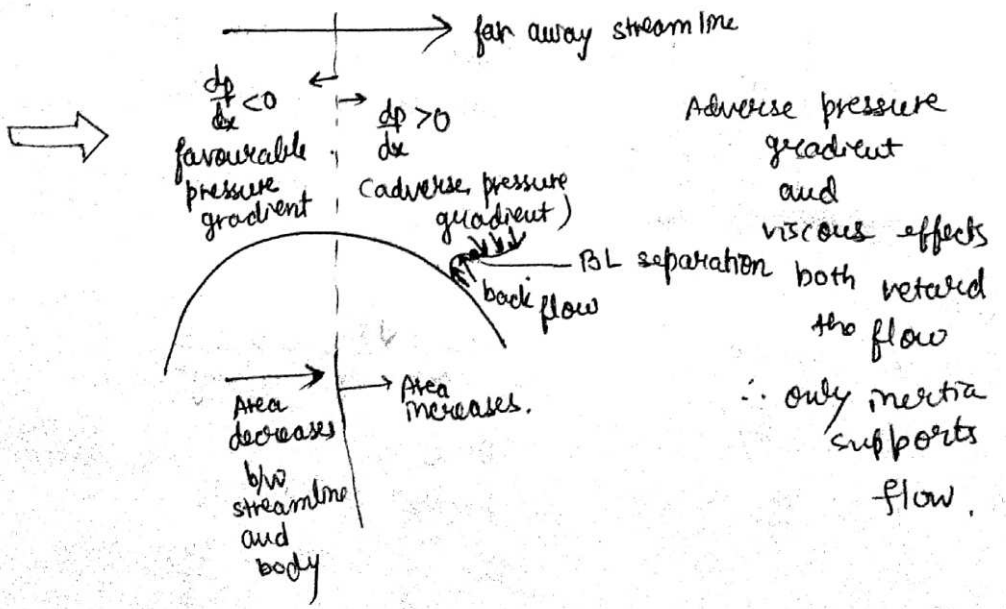
$$\frac{\delta^2}{2}=\frac{3\nu x}{2Au_\infty}+C_1$$

At $x\to 0$, $\delta\to 0 \Rightarrow C_1=0$

$$\frac{\delta}{x}=\sqrt{\frac{3}{A}}\sqrt{\frac{\nu}{u_\infty x}}=()Re_x^{-1/2}$$

$$C_{fx}=\frac{\tau_w}{\frac{1}{2}\rho u_\infty^2}=\frac{\mu\left.\frac{\partial u}{\partial y}\right|_{y=0}}{\frac{1}{2}\rho u_\infty^2}=\frac{\mu\frac{3}{2}\frac{u_\infty}{\delta}}{\frac{1}{2}\rho u_\infty^2}=()Re_x^{-1/2}$$

Boundary Layer with a pressure gradient.



Adverse pressure gradient and viscous effects both retard the flow

∴ only inertia supports flow.

Drag

- Skin friction Drag
- Form Drag / Pressure Drag

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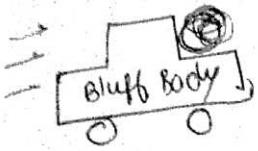
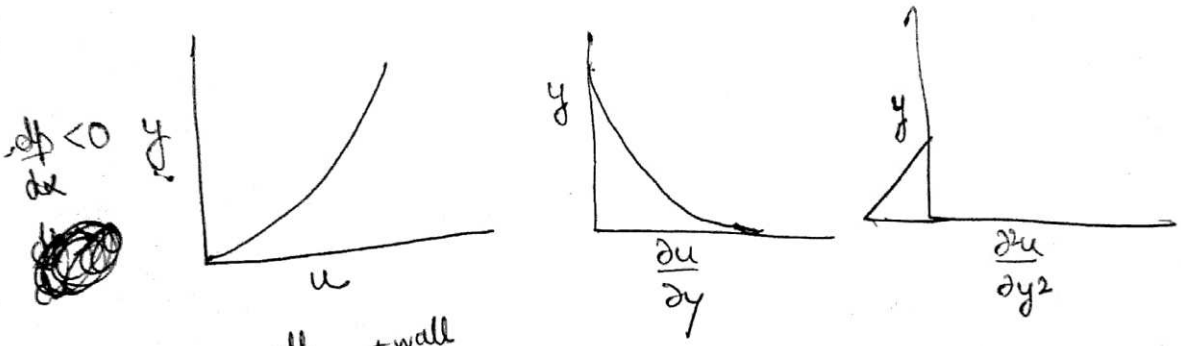
① Streamline body ② Bluff Body



To delay BL separation

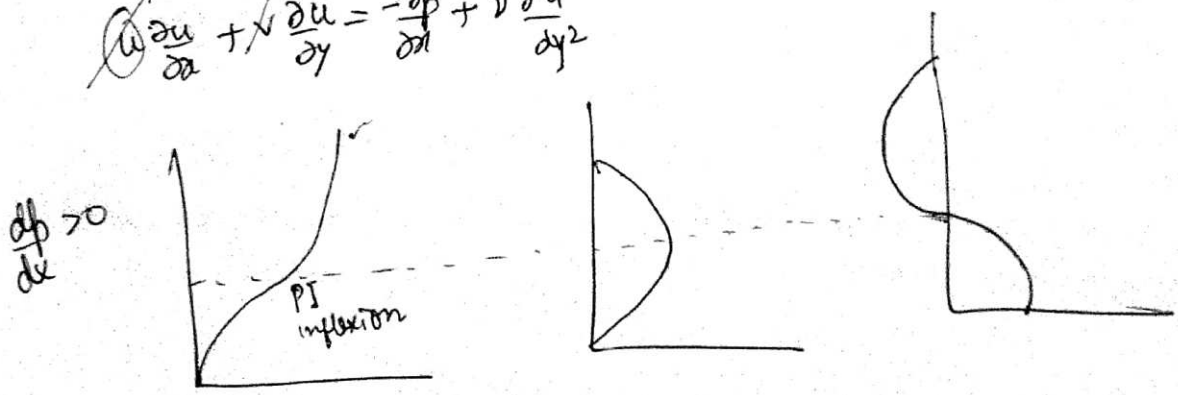
	Laminar Flow	Turbulent Flow
Skin friction drag (Tangential force)	lower	Higher
Form Drag (Normal force)	Higher	Lower

sports ball dynamics – nptel



u 0 at wall, v 0 at wall

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$



$\frac{dp}{dx} = 0$? HW.

Transition to turbulence

Eg. for flow over a flat plate $Re_{cr} \approx 5 \times 10^5$

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$$\frac{\tau_w}{\rho u_\infty^2} = \frac{d}{dx}\int_0^\delta \frac{u}{u_\infty}\left(1-\frac{u}{u_\infty}\right)dy$$

NS & its exact solution ⇒ CT2

Post midsem ⇒ ENDSEM

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