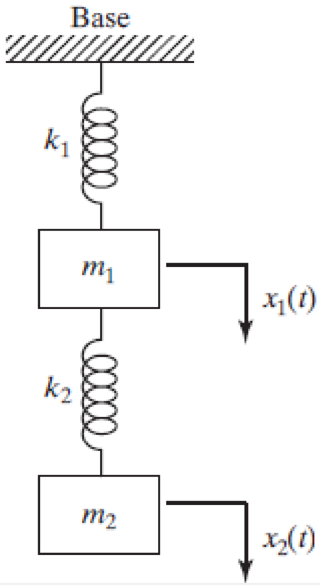
Find the natural frequencies of the system shown in Fig. 5.24, with $m_1 = m$, $m_2 = 2m$, $k_1 = k$, and $k_2 = 2k$. Determine the response of the system when $k = 1000$ N/m, $m = 20$ kg, and the initial values of the displacements of the masses m_1 and m_2 are 1 and -1, respectively.



Equations of motion
$$\left.\begin{aligned} m_1 \ddot{x}_1 + (k_1 + k_2)\, x_1 - k_2 x_2 &= 0 \\ m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 &= 0 \end{aligned}\right\} \quad (E_1)$$

With $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1, 2$, Eqs. (E_1) give the frequency equation

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{vmatrix} = 0$$

or
$$\omega^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right) \omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0 \quad (E_2)$$

Roots of Eq. (E_2) are

$$\omega_1^2, \omega_2^2 = \frac{k_1 + k_2}{2 m_1} + \frac{k_2}{2 m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1 k_2}{m_1 m_2}} \quad (E_3)$$

If $\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} = r_1 X_1^{(1)} \end{Bmatrix}$ and $\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} = r_2 X_1^{(2)} \end{Bmatrix}$,

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1 \omega_1^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2 \omega_1^2 + k_2} \qquad (E_4)$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1 \omega_2^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2 \omega_2^2 + k_2} \qquad (E_5)$$

General solution of (E_1) is

$$x_1(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) \qquad (E_6)$$

$$x_2(t) = r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

where $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 and ϕ_2 can be found using Eqs. (5.18).

For $m_1 = m$, $m_2 = 2m$, $k_1 = k$ and $k_2 = 2k$, (E_3) gives

$$\omega_1^2 = (2 - \sqrt{3}) \frac{k}{m} \quad , \quad \omega_2^2 = (2 + \sqrt{3}) \frac{k}{m} \qquad (E_7)$$

when $k = 1000$ N/m and $m = 20$ kg,

$\omega_1 = 3.6603$ rad/sec and $\omega_2 = 13.6603$ rad/sec

$$r_1 = \frac{k_2}{-m_2 \omega_1^2 + k_2} = 1.36604 , \quad r_2 = \frac{k_2}{-m_2 \omega_2^2 + k_2} = -0.36602$$

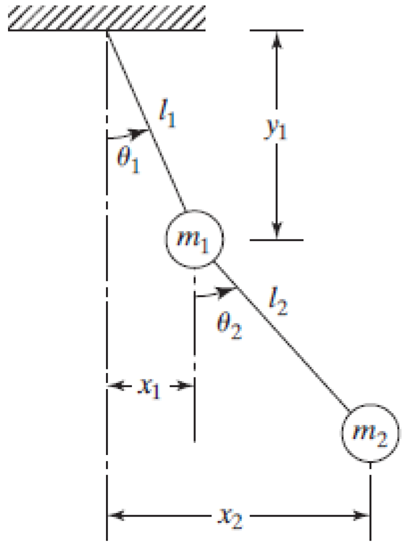
With $x_1(0) = 1$, $\dot{x}_1(0) = 0$, $x_2(0) = -1$ and $\dot{x}_2(0) = 0$, Eqs. (5.18) give $X_1^{(1)} = -0.36602$, $X_1^{(2)} = -1.36603$, $\phi_1 = 0$, $\phi_2 = 0$

Response of the system is

$$x_1(t) = -0.36602 \cos 3.6603 t - 1.36603 \cos 13.6603 t$$

$$x_2(t) = -0.5 \cos 3.6603 t + 0.5 \cos 13.6603 t$$

Set up the differential equations of motion for the double pendulum shown in Fig. 5.25, using the coordinates x_1 and x_2 and assuming small amplitudes. Find the natural frequencies, the ratios of amplitudes, and the locations of nodes for the two modes of vibration when $m_1 = m_2 = m$ and $l_1 = l_2 = l$.

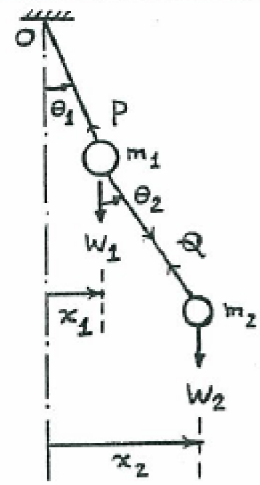


5.6 Taking moments about O and mass m_1,

$$m_1 l_1^2 \ddot{\theta}_1 = -W_1 (l_1 \sin\theta_1) + Q \sin\theta_2 (l_1 \cos\theta_1) - Q\cos\theta_2 (l_1 \sin\theta_1)$$

$$= -W_1 l_1 \theta_1 + W_2 l_1 (\theta_2 - \theta_1) \qquad (E_1)$$

assuming $Q \simeq W_2$.



$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 (l_1 \ddot{\theta}_1) = -W_2 (l_2 \sin\theta_2)$$

$$= -W_2 l_2 \theta_2 \qquad (E_2)$$

Using the relations $\theta_1 = \frac{x_1}{l_1}$ and $\theta_2 = \frac{x_2 - x_1}{l_2}$, Eqs. (E_1) and (E_2) become

$$m_1 l_1 \ddot{x}_1 + \left[W_1 + W_2\left(\frac{l_1 + l_2}{l_2}\right)\right] x_1 - \frac{W_2 l_1}{l_2} x_2 = 0 \qquad (E_3)$$

$$m_2 l_2 \ddot{x}_2 - W_2 x_1 + W_2 x_2 = 0 \qquad (E_4)$$

When $m_1 = m_2 = m$, $l_1 = l_2 = l$ and $W_1 = W_2 = mg$, Eqs. (E_3) and (E_4) give

$$\left.\begin{aligned} m l \ddot{x}_1 + 3mg\, x_1 - mg\, x_2 &= 0 \\ m l \ddot{x}_2 - mg\, x_1 + mg\, x_2 &= 0 \end{aligned}\right\} \qquad (E_5)$$

For harmonic motion $x_i(t) = X_i \cos\omega t$; $i = 1, 2$, Eqs. (E_5) become

become

$$-\omega^2 m l\, X_1 + 3 m g\, X_1 - m g\, X_2 = 0$$
$$-\omega^2 m l\, X_2 - m g\, X_1 + m g\, X_2 = 0 \qquad (E_6)$$

from which the frequency equation can be obtained as

$$\omega^4 m^2 l^2 - (4 m^2 l g)\,\omega^2 + 2\, m^2 g^2 = 0$$

i.e. $$\omega_1^2,\ \omega_2^2 = (2 \mp \sqrt{2})\,\frac{g}{l}$$

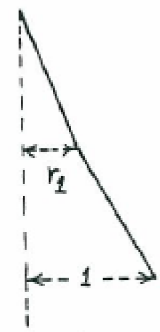
$$\therefore \quad \omega_1 = 0.7654\sqrt{\frac{g}{l}}, \quad \omega_2 = 1.8478\sqrt{\frac{g}{l}}$$

Ratio of amplitudes is given by Eq. (E_6) as

$$\frac{X_1}{X_2} = \frac{mg}{-\omega^2 m l + 3mg} = \frac{1}{\left(-\omega^2 \frac{l}{g} + 3\right)}$$

In mode 1, $\omega_1 = 0.7654\sqrt{\frac{g}{l}}$, $r_1 = \left(\frac{X_1}{X_2}\right)^{(1)} = 0.4142$

No node.



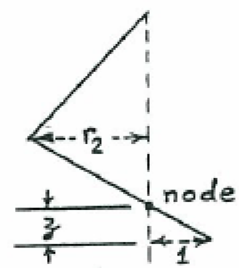
mode 1

In mode 2, $\omega_2 = 1.8478\sqrt{\frac{g}{l}}$,

$$r_2 = \left(\frac{X_1}{X_2}\right)^{(2)} = -2.4133$$

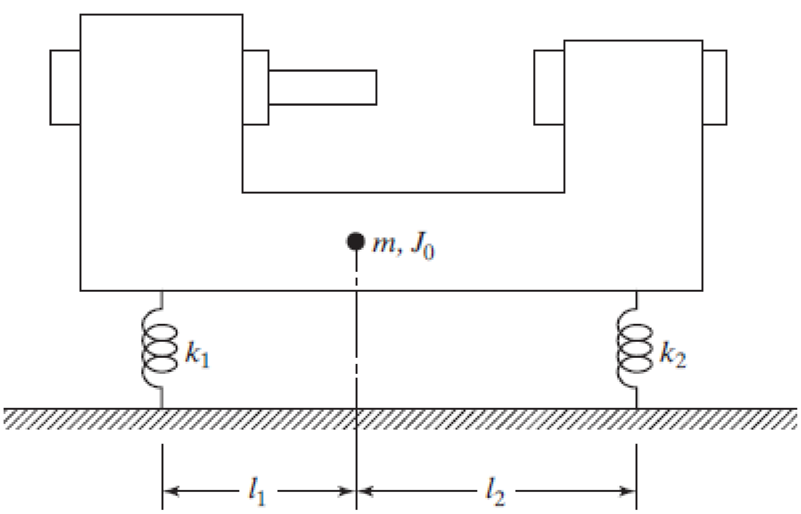
one node located at z:

$$\frac{z}{1} = \frac{1-z}{2.4133} \quad \text{or} \quad z = 0.2930$$

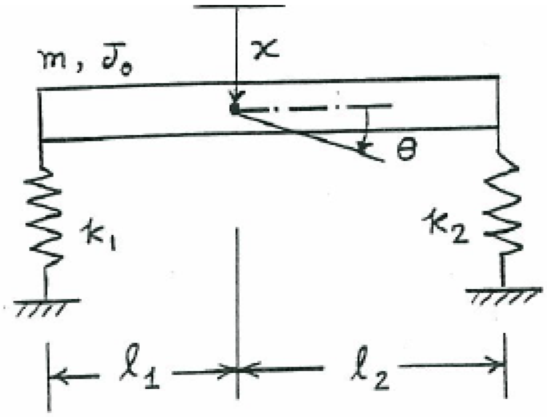


mode 2

A machine tool, having a mass of $m = 1000$ kg and a mass moment of inertia of $J_0 = 300$ kg-m^2, is supported on elastic supports, as shown in Fig. 5.27. If the stiffnesses of the supports are given by $k_1 = 3000$ N/mm and $k_2 = 2000$ N/mm, and the supports are located at $l_1 = 0.5$ m and $l_2 = 0.8$ m, find the natural frequencies and mode shapes of the machine tool.



Equations of motion in terms of x and θ:



$$m\ddot{x} + k_1(x - l_1\theta) + k_2(x + l_2\theta) = 0 \qquad (E_1)$$

$$J_0\ddot{\theta} - k_1 l_1 (x - l_1\theta) + k_2 l_2 (x + l_2\theta) = 0 \qquad (E_2)$$

For free vibration,

$$x(t) = X \cos(\omega t + \phi) \qquad (E_3)$$

$$\theta(t) = \Theta \cos(\omega t + \phi) \qquad (E_4)$$

and Eqs. (E_1) and (E_2) become

$$\begin{bmatrix} -m\omega^2 + k_1 + k_2 & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & -J_0\omega^2 + k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (E_5)$$

Frequency equation is

$$\begin{vmatrix} -m\omega^2 + k_1 + k_2 & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & -J_0\omega^2 + k_1 l_1^2 + k_2 l_2^2 \end{vmatrix} = 0 \qquad (E_6)$$

i.e.,

$$\begin{vmatrix} -\omega^2 + 5000 & 100 \\ 100 & -0.3\,\omega^2 + 2030 \end{vmatrix} = 0$$

i.e.,

$$0.3\,\omega^4 - 3530\,\omega^2 + 10.14 \times 10^6 = 0$$

i.e., $\omega^2 = 6785.3373, \quad 4981.3293$

$\therefore \quad \omega_1 = 70.5785$ rad/sec , $\quad \omega_2 = 82.3732$ rad/sec

Mode shapes:

$$(-1000\,\omega_1^2 + 5 \times 10^6)\,X + 0.1 \times 10^6\,\Theta = 0$$

or

$$\left.\frac{X}{\Theta}\right|_{\omega_1} = \frac{-0.1 \times 10^6}{-1000\,\omega_1^2 + 5 \times 10^6} = -5.3476$$

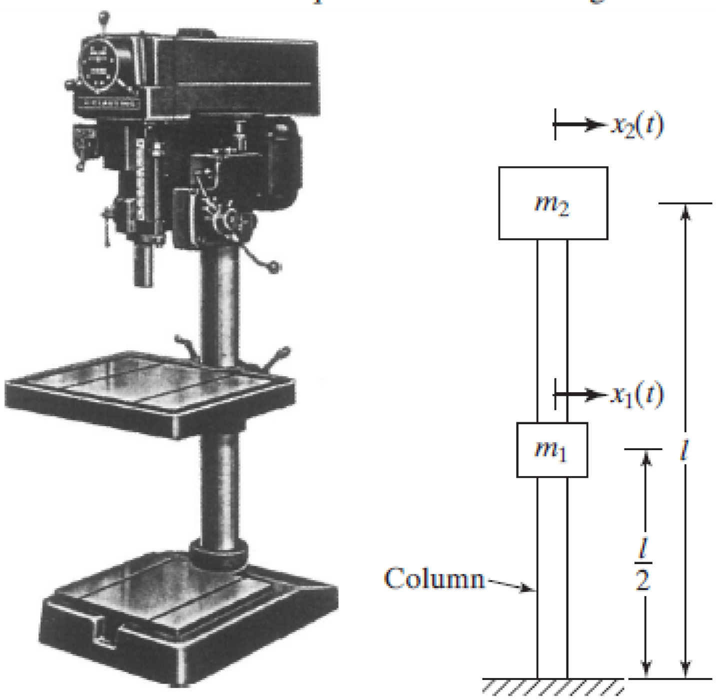
and

$$\left.\frac{X}{\Theta}\right|_{\omega_2} = \frac{-0.1 \times 10^6}{-1000\,\omega_2^2 + 5 \times 10^6} = 0.05601$$

The drilling machine shown in Fig. 5.29(a) can be modeled as a two-degree-of-freedom system as indicated in Fig. 5.29(b). Since a transverse force applied to mass m_1 or mass m_2 causes both the masses to deflect, the system exhibits elastic coupling. The bending stiffnesses of the column are given by (see Section 6.4 for the definition of stiffness influence coefficients)

$$k_{11} = \frac{768}{7}\frac{EI}{l^3}, \qquad k_{12} = k_{21} = -\frac{240}{7}\frac{EI}{l^3}, \qquad k_{22} = \frac{96}{7}\frac{EI}{l^3}$$

Determine the natural frequencies of the drilling machine.



(a) (b)

FIGURE 5.29 (Photo courtesy of Atlas-Clausing

Frequency equation:

$$\left| \left[-\omega^2 \, [m] + [k] \right] \right| = 0$$

or

$$\begin{vmatrix} (k_{11} - \omega^2 \, m_1) & k_{12} \\ k_{21} & (k_{22} - \omega^2 \, m_2) \end{vmatrix} = 0 \qquad (1)$$

Expansion of the determinantal equation (1) gives:

$$(m_1 \, m_2) \, \omega^4 - (m_1 \, k_{22} + m_2 \, k_{11}) \, \omega^2 + (k_{11} \, k_{22} - k_{12}^2) = 0 \qquad (2)$$

Roots of Eq. (2):

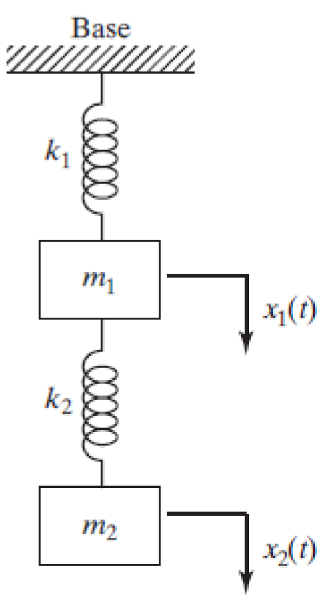
$$\omega_2^2, \, \omega_1^2 = \frac{(m_1 \, k_{22} + m_2 \, k_{11}) \pm \sqrt{\left\{ (m_1 \, k_{22} - m_2 \, k_{11})^2 + 4 \, m_1 \, m_2 \, k_{12}^2 \right\}}}{2 \, m_1 \, m_2} \qquad (3)$$

Substitution of known expressions for k_{11}, k_{12}, and k_{22} into Eq. (3) yields:

$$\omega_2^2, \, \omega_1^2 = \frac{48}{7} \, \frac{E \, I}{m_1 \, m_2} \left[(m_1 + 8 \, m_2) \pm \sqrt{(m_1 - 8 \, m_2)^2 + 25 \, m_1 \, m_2} \right] \qquad (4)$$

===

Determine the initial conditions of the system shown in Fig. 5.24 for which the system vibrates only at its lowest natural frequency for the following data: $k_1 = k$, $k_2 = 2k$, $m_1 = m$, $m_2 = 2m$.



5.25 From solution of Problem 5.5, we find that for $m_1 = m$, $m_2 = 2\ m$, $k_1 = k$ and $k_2 = 2\ k$,

$$\omega_1^2 = (2 - \sqrt{3})\ \frac{k}{m} \quad ; \quad \omega_2^2 = (2 + \sqrt{3})\ \frac{k}{m}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{k_2}{-\ m_2\ \omega_1^2 + k_2} = \frac{1}{-1 + \sqrt{3}}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{k_2}{-\ m_2\ \omega_2^2 + k_2} = \frac{1}{-1 - \sqrt{3}}$$

First mode shape:

$$\begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)}\ \cos(\omega_1\ t + \phi_1) \\ r_1\ X_1^{(1)}\ \cos(\omega_1\ t + \phi_1) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)}\ \cos(\omega_1\ t + \phi_1) \\ \left(\dfrac{X_1^{(1)}}{\sqrt{3} - 1}\right)\ \cos(\omega_1\ t + \phi_1) \end{Bmatrix}$$

For the motion to be identical with the first normal mode, we need to have $X_1^{(2)} = 0$. This requires that (from Eq. (5.18)):

$$\frac{1}{r_2 - r_1}\left[\left\{-\ r_1\ x_1(0) + x_2(0)\right\}^2 + \frac{1}{\omega_2^2}\left\{r_1\ \dot{x}_1(0) - \dot{x}_2(0)\right\}^2\right]^{\frac{1}{2}} = 0$$

or

$$x_2(0) = r_1\ x_1(0) = \frac{x_1(0)}{\sqrt{3} - 1}$$

$$\dot{x}_2(0) = r_1\ \dot{x}_1(0) = \frac{\dot{x}_1(0)}{\sqrt{3} - 1}$$

===

The system shown in Fig. 5.24 is initially disturbed by holding the mass m_1 stationary and giving the mass m_2 a downward displacement of 0.1 m. Discuss the nature of the resulting motion of the system.

Let $m_1 = m$, $m_2 = 2\,m$, $k_1 = k$, $k_2 = 2\,k$.
Initial conditions: $x_1(0) = 0$, $x_2(0) = 0.1$ m, $\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0$
Eqs. (5.18) yield:

$$X_1^{(1)} = \frac{1}{r_2 - r_1}\left[(0 - 0.1)^2\right]^{\frac{1}{2}} = \frac{0.1}{r_2 - r_1} = \frac{0.1}{\left(\frac{-1}{\sqrt{3}+1} - \frac{1}{\sqrt{3}-1}\right)} = -\frac{0.1}{\sqrt{3}}$$

$$X_1^{(2)} = \frac{1}{r_2 - r_1}\left[(0.1)^2\right]^{\frac{1}{2}} = \frac{0.1}{r_2 - r_1} = -\frac{0.1}{\sqrt{3}}$$

$$\phi_1 = \tan^{-1}(0) = 0$$

$$\phi_2 = \tan^{-1}(0) = 0$$

where ω_1 and ω_2 are given by Eq. (E3) of solution of Problem 5.5.

Resulting motion:

$$x_1(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) = -\frac{0.1}{\sqrt{3}}\left\{\cos \omega_1 t + \cos \omega_2 t\right\}$$

$$x_2(t) = r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$= \left(\frac{1}{\sqrt{3}-1}\right)\left(-\frac{0.1}{\sqrt{3}}\right)\cos \omega_1 t + \left(-\frac{1}{\sqrt{3}+1}\right)\left(-\frac{0.1}{\sqrt{3}}\right)\cos \omega_2 t$$

$$= -\frac{0.1}{\sqrt{3}}\left[\left(\frac{1}{\sqrt{3}-1}\right)\cos \omega_1 t - \left(\frac{1}{\sqrt{3}+1}\right)\cos \omega_2 t\right]$$

===

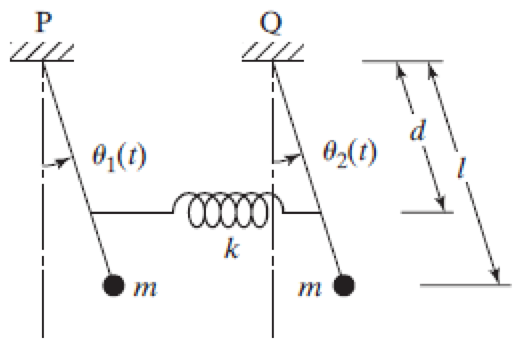
Two identical pendulums, each with mass m and length l, are connected by a spring of stiffness k at a distance d from the fixed end, as shown in Fig. 5.35.

a. Derive the equations of motion of the two masses.

b. Find the natural frequencies and mode shapes of the system.

c. Find the free-vibration response of the system for the initial conditions $\theta_1(0) = a$, $\theta_2(0) = 0$, $\dot{\theta}_1(0) = 0$, and $\dot{\theta}_2(0) = 0$.

d. Determine the condition(s) under which the system exhibits a beating phenomenon.



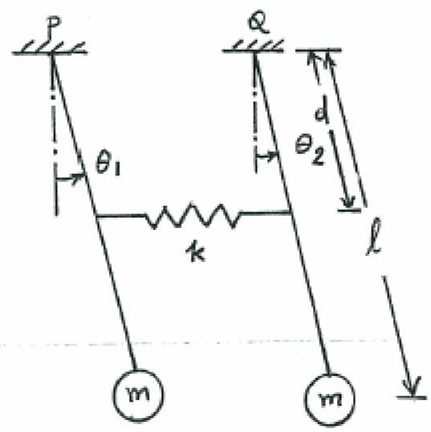
5.31 (a) Equations of motion:

Assume: θ_1, θ_2 are small.

Moment equilibrium equations of the two masses about P and Q:

$$m l^2 \ddot{\theta}_1 + m g l \theta_1 + k d^2 (\theta_1 - \theta_2) = 0 \qquad (1)$$

$$m l^2 \ddot{\theta}_2 + m g l \theta_2 - k d^2 (\theta_1 - \theta_2) = 0 \qquad (2)$$

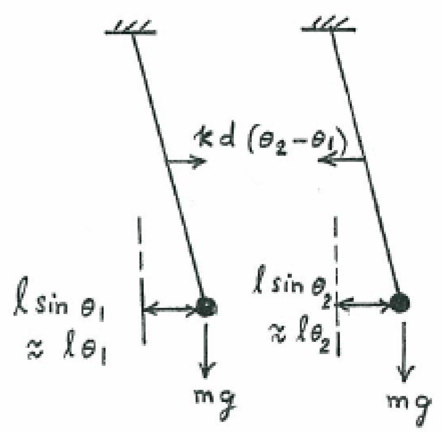


(b) Natural frequencies and mode shapes:

Assume: Harmonic motion with

$$\theta_i(t) = \Theta_i \cos(\omega t - \phi); \quad i = 1, 2 \qquad (3)$$

where Θ_1 and Θ_2 are amplitudes of θ_1 and θ_2, respectively, ω is the natural frequency, and ϕ is the phase angle.



Free body diagram

Using Eq. (3), Eqs. (1) and (2) can be expressed in matrix form as

$$-\omega^2 m \ell^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} + \begin{bmatrix} mg\ell + kd^2 & -kd^2 \\ -kd^2 & mg\ell + kd^2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (4)$$

Frequency equation:

$$\begin{vmatrix} -\omega^2 m \ell^2 + mg\ell + kd^2 & -kd^2 \\ -kd^2 & -\omega^2 m\ell^2 + mg\ell + kd^2 \end{vmatrix} = 0$$

or

$$\omega^4 - \omega^2 \left(\frac{2g}{\ell} + \frac{2kd^2}{m\ell^2} \right) + \left(\frac{g^2}{\ell^2} + \frac{2gkd^2}{m\ell^3} \right) = 0 \qquad (5)$$

Solution of Eq. (5) gives

$$\omega_1^2 = \frac{g}{\ell}, \qquad \omega_2^2 = \frac{g}{\ell} + \frac{2kd^2}{m\ell^2} \qquad (6)$$

By substituting for ω_1^2 and ω_2^2 into Eq. (4), we obtain

$$\left(\frac{\Theta_2}{\Theta_1} \right)^{(1)} = 1 \quad \text{or} \quad \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \Theta_1^{(1)}$$

and

$$\left(\frac{\Theta_2}{\Theta_1} \right)^{(2)} = -1 \quad \text{or} \quad \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \Theta_1^{(2)}$$

Thus the motion of the masses in the two modes is given by

$$\vec{\theta}^{(1)}(t) = \begin{Bmatrix} \theta_1^{(1)}(t) \\ \theta_2^{(1)}(t) \end{Bmatrix} = \Theta_1^{(1)} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos(\omega_1 t + \phi_1) \qquad (7)$$

$$\vec{\theta}^{(2)}(t) = \begin{Bmatrix} \theta_1^{(2)}(t) \\ \theta_2^{(2)}(t) \end{Bmatrix} = \Theta_1^{(2)} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \cos(\omega_2 t + \phi_2) \qquad (8)$$

(c) Free vibration response:

Using linear superposition of natural modes, the free vibration response of the system is given by

$$\vec{\theta}(t) = c_1 \vec{\theta}^{(1)}(t) + c_2 \vec{\theta}^{(2)}(t) \qquad (9)$$

By choosing $c_1 = c_2 = 1$, with no loss of generality, Eqs. (7) to (9) lead to

$$\theta_1(t) = \Theta_1^{(1)} \cos(\omega_1 t + \phi_1) + \Theta_1^{(2)} \cos(\omega_2 t + \phi_2) \qquad (10)$$

$$\theta_2(t) = \Theta_1^{(1)} \cos(\omega_1 t + \phi_1) - \Theta_1^{(2)} \cos(\omega_2 t + \phi_2) \qquad (11)$$

where $\Theta_1^{(1)}$, ϕ_1, $\Theta_1^{(2)}$ and ϕ_2 are constants to be determined from the initial conditions. When $\theta_1(0) = \alpha$, $\theta_2(0) = 0$, $\dot{\theta}_1(0) = 0$ and $\dot{\theta}_2(0) = 0$, Eqs. (10) and (11) yield

$$\left.\begin{aligned} \alpha &= \Theta_1^{(1)} \cos\phi_1 + \Theta_1^{(2)} \cos\phi_2 \\ 0 &= \Theta_1^{(1)} \cos\phi_1 - \Theta_1^{(2)} \cos\phi_2 \\ 0 &= -\omega_1 \Theta_1^{(1)} \sin\phi_1 - \omega_2 \Theta_1^{(2)} \sin\phi_2 \\ 0 &= -\omega_1 \Theta_1^{(1)} \sin\phi_1 + \omega_2 \Theta_1^{(2)} \sin\phi_2 \end{aligned}\right\} \qquad (12)$$

Eqs. (12) can be solved for $\Theta_1^{(1)}$, ϕ_1, $\Theta_1^{(2)}$ and ϕ_2 to obtain

$$\left.\begin{aligned} \theta_1(t) &= \alpha \cos\frac{\omega_2 - \omega_1}{2} t \cdot \cos\frac{\omega_2 + \omega_1}{2} t \\ \theta_2(t) &= \alpha \sin\frac{\omega_2 - \omega_1}{2} t \cdot \sin\frac{\omega_2 + \omega_1}{2} t \end{aligned}\right\} \qquad (13)$$

<u>(d) conditions for beating:</u>

$$\text{When} \quad \frac{2kd^2}{ml^2} \ll \frac{g}{l} \quad \text{or} \quad k \ll \frac{mgl}{2d^2}, \qquad (14)$$

the two frequency components in Eqs. (13), namely, $\frac{\omega_2 - \omega_1}{2}$ and $\frac{\omega_2 + \omega_1}{2}$, can be approximated as

$$\Omega_1 = \frac{\omega_2 - \omega_1}{2} \simeq \frac{k}{2m} \frac{d^2}{\sqrt{gl^3}} \qquad (15)$$

and

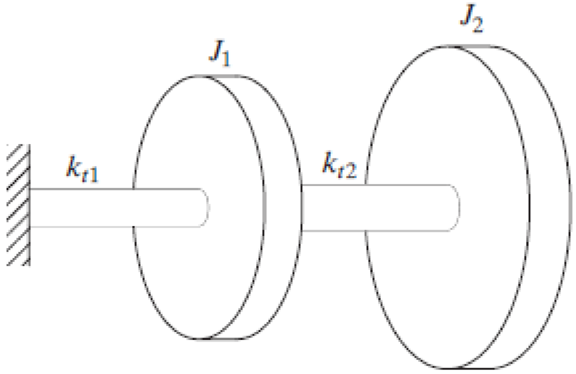
$$\Omega_2 = \frac{\omega_2 + \omega_1}{2} \simeq \sqrt{\frac{g}{l}} + \frac{k}{2m} \frac{d^2}{\sqrt{gl^3}} \qquad (16)$$

This implies that the motions of the pendulums are given by

$$\left.\begin{aligned} \theta_1(t) &\simeq \alpha \cos\Omega_1 t \cdot \cos\Omega_2 t \\ \theta_2(t) &\simeq \alpha \sin\Omega_1 t \cdot \sin\Omega_2 t \end{aligned}\right\} \qquad (17)$$

This motion, Eqs. (17), denotes beating phenomenon.

5.36 Determine the natural frequencies and normal modes of the torsional system shown in Fig. 5.38 for $k_{t2} = 2k_{t1}$ and $J_2 = 2J_1$.



With $k_{t1} = k_t$, $k_{t2} = 2k_t$, $J_1 = J_0$, $J_2 = 2J_0$, $k_{t3} = 0$ and $M_{t1} = M_{t2} = 0$, Eqs. (5.20) give

$$J_0\ddot{\theta}_1 + 3k_t\theta_1 - 2k_t\theta_2 = 0$$

$$2J_0\ddot{\theta}_2 - 2k_t\theta_1 + 2k_t\theta_2 = 0$$

For harmonic solution, $\theta_i(t) = \Theta_i \cos(\omega t + \phi)$; $i = 1, 2$,

$$\begin{bmatrix} (-\omega^2 J_0 + 3k_t) & -2k_t \\ -2k_t & (-2\omega^2 J_0 + 2k_t) \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

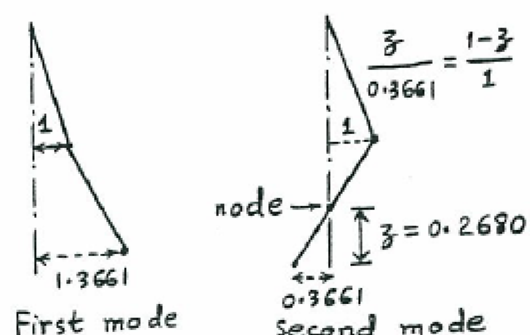
Frequency equation is

$$\begin{vmatrix} -\omega^2 J_0 + 3k_t & -2k_t \\ -2k_t & -2\omega^2 J_0 + 2k_t \end{vmatrix} = 2J_0^2\omega^4 - 8J_0 k_t \omega^2 + 2k_t^2 = 0$$

$$\omega^2 = (2 \mp \sqrt{3})\frac{k_t}{J_0} \quad ; \quad \omega_1 = 0.5176\sqrt{\frac{k_t}{J_0}}, \quad \omega_2 = 1.9319\sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-J_0\omega_1^2 + 3k_t}{2k_t} = 1.3661$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-J_0\omega_2^2 + 3k_t}{2k_t} = -0.3661$$



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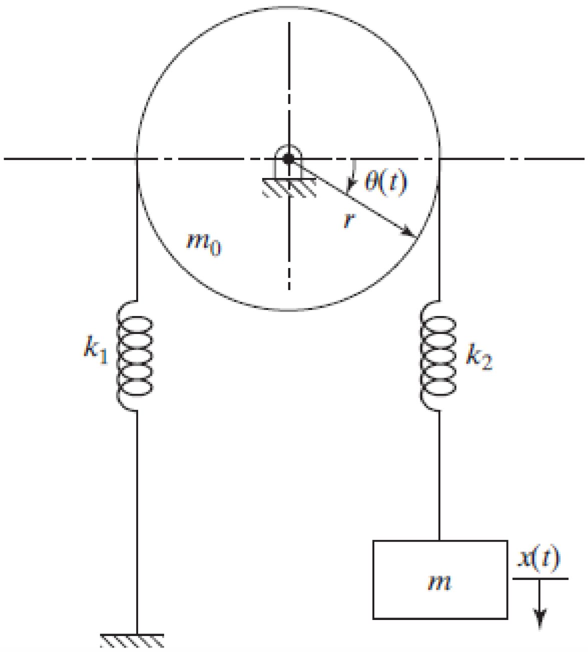
5.37 Determine the natural frequencies of the system shown in Fig. 5.39 by assuming that the rope passing over the cylinder does not slip.



5.37 Equation of motion of mass m: $\quad m\ddot{x} = -k_2\,(x - r\theta)$ ---(E_1)

Equation of motion of cylinder of mass m_0 and mass moment of inertia $J_0 = \frac{1}{2} m_0 r^2$:

$$J_0\,\ddot{\theta} = -k_1 r^2 \theta - k_2\,(r\theta - x)\,r \quad \text{---}(E_2)$$

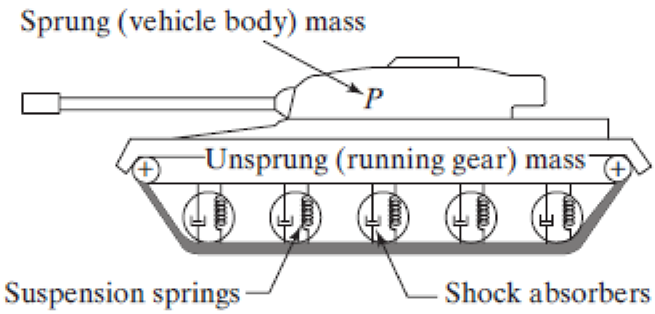
For $x(t) = X\cos(\omega t + \phi)$ and $\theta(t) = \Theta\cos(\omega t + \phi)$, Eqs. (E_1) and (E_2) give the frequency equation

$$\begin{vmatrix} -m\omega^2 + k_2 & -k_2 r \\ -k_2 r & -\frac{1}{2} m_0 r^2 \omega^2 + k_1 r^2 + k_2 r^2 \end{vmatrix} = 0$$

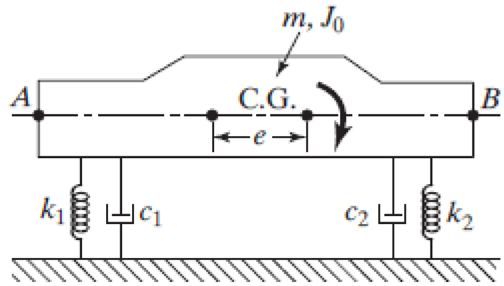
i.e. $$\omega^4 - \omega^2\left(\frac{k_2}{m} + \frac{2\{k_1 + k_2\}}{m_0}\right) + \frac{2\,k_1 k_2}{m_0\, m} = 0$$

$$\omega_1^2,\ \omega_2^2 = \frac{k_2}{2m} + \frac{(k_1 + k_2)}{m_0} \mp \sqrt{\frac{1}{4}\left(\frac{k_2}{m} + \frac{2\,k_1}{m_0} + \frac{2\,k_2}{m_0}\right)^2 - \frac{2\,k_1 k_2}{m\, m_0}}$$

5.40 A simplified ride model of the military vehicle in Fig. 5.40(a) is shown in Fig. 5.40(b). This model can be used to obtain information about the bounce and pitch modes of the vehicle. If the total mass of the vehicle is m and the mass moment of inertia about its C.G. is J_0, derive the equations of motion of the vehicle using two different sets of coordinates, as indicated in Section 5.5.



(a) Military vehicle

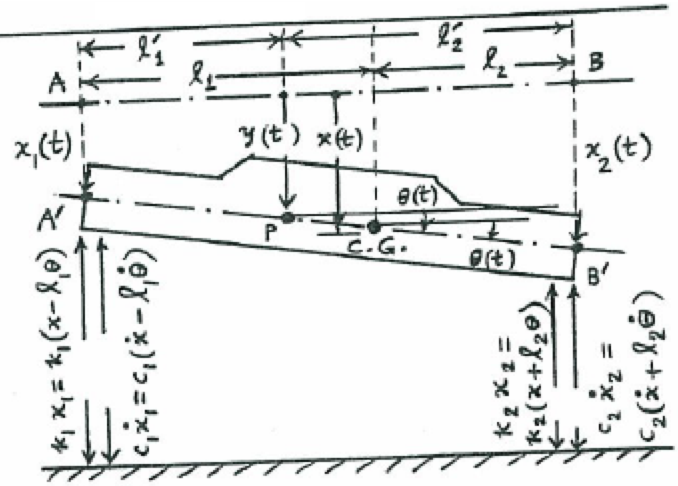


(b) Simplified ride model

(i) Using $x(t)$ and $\theta(t)$:

For translatory motion:

$$m\ddot{x} = -k_1(x-l_1\theta) - c_1(\dot{x}-l_1\dot{\theta}) - k_2(x+l_2\theta) - c_2(\dot{x}+l_2\dot{\theta}) \quad \text{--- } (E_1)$$



For rotational motion about C.G.:

$$J_0\ddot{\theta} = k_1(x-l_1\theta)l_1 + c_1(\dot{x}-l_1\dot{\theta})l_1 - k_2(x+l_2\theta)l_2 - c_2(\dot{x}+l_2\dot{\theta})l_2 \quad \text{--- } (E_2)$$

Eqs. (E_1) and (E_2) can be rewritten as

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_1 l_1 + c_2 l_2 \\ -c_1 l_1 + c_2 l_2 & c_1 l_1^2 + c_2 l_2^2 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_1 l_1 + k_2 l_2 \\ -k_1 l_1 + k_2 l_2 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(ii) Using $y(t)$ and $\theta(t)$:

For translatory motion:

$$m\ddot{y} = -k_1(y - l_1'\theta) - c_1(\dot{y} - l_1'\dot{\theta}) - k_2(y + l_2'\theta) - c_2(\dot{y} + l_2'\dot{\theta}) - me\ddot{\theta} \quad \text{---- } (E_3)$$

For rotational motion:

$$J_p\ddot{\theta} = k_1(y - l_1'\theta)l_1' + c_1(\dot{y} - l_1'\dot{\theta})l_1' - k_2(y + l_2'\theta)l_2' - c_2(\dot{y} + l_2'\dot{\theta})l_2' - me\ddot{y} \quad \text{---- } (E_4)$$

Eqs. (E_3) and (E_4) can be rewritten as

$$\begin{bmatrix} m & me \\ me & J_p \end{bmatrix} \begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_1 l_1' + c_2 l_2' \\ -c_1 l_1' + c_2 l_2' & c_1 l_1'^2 + c_2 l_2'^2 \end{bmatrix} \begin{Bmatrix} \dot{y} \\ \dot{\theta} \end{Bmatrix}$$

$$+ \begin{bmatrix} k_1 + k_2 & -k_1 l_1' + k_2 l_2' \\ -k_1 l_1' + k_2 l_2' & k_1 l_1'^2 + k_2 l_2'^2 \end{bmatrix} \begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

A trailer of mass M is connected to a wall through a spring of stiffness k_1 and can move on a frictionless horizontal surface, as shown in Fig. 5.50. A uniform cylinder of mass m, connected to the wall of the trailer by a spring of stiffness k_2, can roll on the floor of the trailer without slipping. Derive the equations of motion of the system and discuss the nature of coupling present in the system.

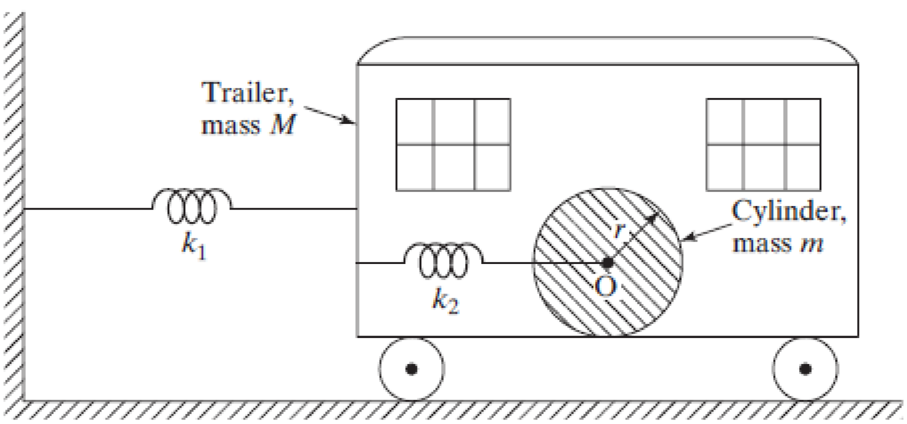
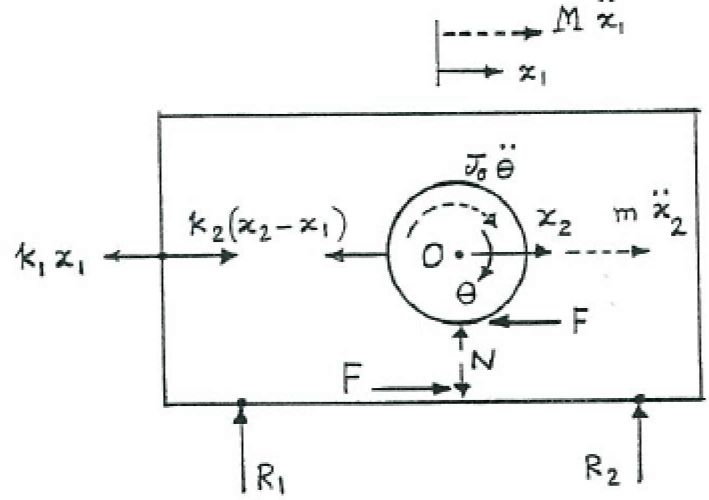


FIGURE 5.50

1)



Free body diagram

N = normal reaction between cylinder and trailer, F = friction force, R_1, R_2 = reactions between trailer and ground.

Equation of motion for linear motion of cylinder:

$$\sum F = m\ddot{x}_2 \quad \text{or} \quad m\ddot{x}_2 = -F - k_2(x_2 - x_1) \tag{1}$$

Equation of motion for rotational motion of cylinder:

$$\sum M_0 = J_0\ddot{\theta} \quad \text{or} \quad J_0\ddot{\theta} = F\,r \tag{2}$$

where $J_0 = \frac{1}{2}\,m\,r^2$ and $\theta = \frac{x_2 - x_1}{r}$.

Equation of motion for linear motion of trailer:

$$\sum F = M\ddot{x}_1 \quad \text{or} \quad M\ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) + F \tag{3}$$

Eq. (2) gives

$$F = \frac{J_0\ddot{\theta}}{r} = \frac{1}{r}\left(\frac{1}{2}\,m\,r^2\right)\left(\frac{\ddot{x}_2 - \ddot{x}_1}{r}\right) = \frac{m}{2}(\ddot{x}_2 - \ddot{x}_1) \tag{4}$$

Substitution of Eq. (4) into Eqs. (1) and (3) yields the equations of motion as:

$$\frac{3\,m}{2}\ddot{x}_2 - \frac{1}{2}\,m\,\ddot{x}_1 - k_2 x_1 + k_2 x_2 = 0 \tag{5}$$

$$\left(M + \frac{m}{2}\right)\ddot{x}_1 - \frac{m}{2}\ddot{x}_2 + x_1(k_1 + k_2) - k_2 x_2 = 0 \tag{6}$$

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