1.2.17

Design of Machine Elements – Lecture 14

STRESS ANALYSIS

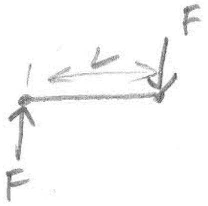
Properties of (Cauchy) stress : Stress defined in deformed/current config.

- It is a second-order tensor and has 9 components denoted by σ_{ij} $(i, j = 1, 2, 3)$

- The 9 components can be denoted in matrix form as

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

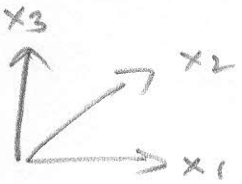
- In absence of point couples (

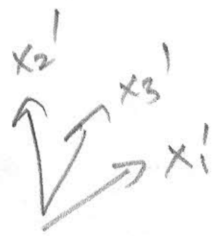


$C = FL$, with $F \to \infty$ and $L \to 0$) – which is the case in essentially all engineering problems – angular momentum balance requires that stress tensor be symmetric and $\boxed{\sigma_{ij} = \sigma_{ji}}$

- Stress components depend on choice of co-ordinate system. However, components of stress in different co-ordinate systems are related through transformation relations.

More specifically, if $x_1 - x_2 - x_3$ and





$x_1' - x_2' - x_3'$ denote two rectangular cartesian co-ordinate systems with $\underset{\sim}{e}'_i$ and $\underset{\sim}{e}_j$ denoting unit vectors along x'_i and x_j $(i, j = 1, 2, 3)$ axes then

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{31} & Q_{32} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}^T$$

where σ'_{ij} are components of stress in x'_i co-ordinate system σ_{ij} are components of stress in x_i system and Q_{ij} are components of rotation matrix given by

$$\boxed{Q_{ij} = \underset{\sim}{e}'_i \cdot \underset{\sim}{e}_j}$$

with

$$[Q][Q]^T = [I] \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det[Q] = \det[Q]^T = 1$$

- While components of stress depend on co-ordinate system certain "measures" of stress are independent of co-ordinate system. Such "measures" are called stress invariants

- Noting that for any real matrices $[A], [B], [C]$

$$tr([A][B][C]) = tr([C][A][B]) = tr([B][C][A])$$

and $$\det([A][B][C]) = \det[A] \det[B] \det[C]$$

we have that

$$tr([\sigma']) = tr([Q][\sigma][Q]^T)$$

$$= tr([Q]^T[Q][\sigma]) = tr([I][\sigma])$$

$$\Rightarrow \boxed{tr([\sigma']) = tr([\sigma])}$$

Thus trace of stress $= \sigma_{11}+\sigma_{22}+\sigma_{33}$ $= -3p$ is a stress invariant

Similarly

$$det([\sigma']) = det[Q] \; det[\sigma] \; det[Q]^T$$

$$\Rightarrow \boxed{det[\sigma'] = det[\sigma]}$$

Thus "$det[\sigma]$" is a stress invariant

$$tr([\sigma']^2) = tr([Q][\sigma][Q]^T[Q][\sigma][Q]^T)$$

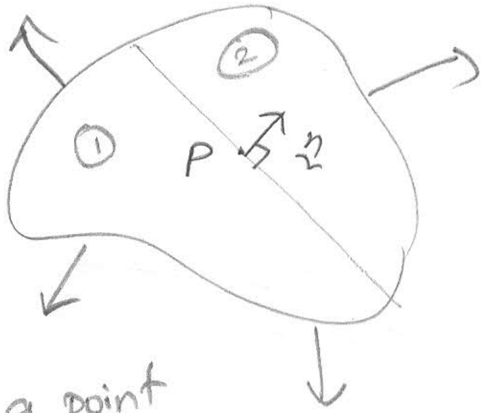
$$= tr([Q][\sigma]^2[Q]^T) = tr([Q]^T[Q][\sigma]^2)$$

$$\Rightarrow \boxed{tr([\sigma']^2) = tr([\sigma]^2)}$$

"$tr[\sigma]^2$" is another stress invariant

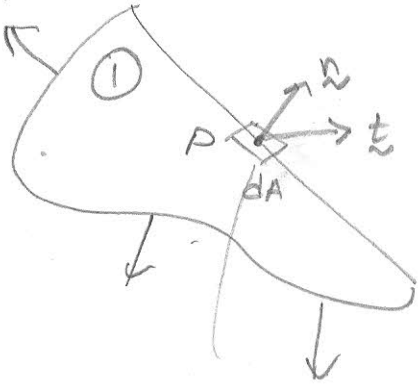
- Stress exists at all points within a body and is commonly non-uniform, i.e. different at different points.

- Consider a body subjected to external loads that results in "stress" at all points in the body.



Let $[\sigma]$ denote stress at a point "P" and let $\underset{\sim}{n}$ denote unit normal to a plane passing through P.

Force acting per unit area $\underset{\sim}{t}$ at P on plane with unit normal $\underset{\sim}{n}$ is given by



$$\underset{\sim}{t} = \underset{\approx}{\sigma}^T \underset{\sim}{n}$$

or

$$\boxed{\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}^T \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}}$$

Force acting on area element dA on plane in question is

$$d\underset{\sim}{F} = \underset{\sim}{t}\, dA$$

passing through a point

- Planes on which ONLY normal traction exists and shear traction is zero, i.e.

$$\boxed{\underset{\sim}{t}(\underset{\sim}{n}) = \lambda \underset{\sim}{n} \quad \text{or} \quad \{t\} = \underbrace{\lambda\{n\}}_{\text{principal stress}}}$$

are called principal planes and normal tractions acting on these planes are called principal stresses

- Principal stresses are determined from characteristic eq. given by

$$[\sigma]\{n\} = \lambda\{n\}$$

$$\Rightarrow \quad ([\sigma] - \lambda[I])\{n\} = 0$$

$$\Rightarrow \quad \det([\sigma] - \lambda[I]) = 0$$

$$\Rightarrow \quad \begin{vmatrix} \sigma_{11}-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\lambda & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \boxed{-\lambda^3 + (\underbrace{\sigma_{11}+\sigma_{22}+\sigma_{33}}_{\equiv \mathrm{tr}[\sigma]})\lambda^2 - (\underbrace{\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}-\sigma_{12}^2-\sigma_{13}^2-\sigma_{23}^2}_{\equiv \frac{1}{2}\left((\mathrm{tr}[\sigma])^2 - \mathrm{tr}[\sigma]^2\right)})\lambda + \det[\sigma] = 0}$$

$$\Rightarrow \boxed{-\lambda^3 + \mathrm{tr}[\sigma]\lambda^2 - \frac{1}{2}\left((\mathrm{tr}[\sigma])^2 - \mathrm{tr}[\sigma]^2\right)\lambda + \det[\sigma] = 0}$$

Characteristic eq. for principal stresses

Coefficients of characteristic eq. are independent of co-ordinate system, thus, principal stresses are also co-ordinate invariants.

- In any problem at any point there is at least ONE set of mutually perpendicular principal planes

- If $\sigma_1 \geq \sigma_2 \geq \sigma_3$ denote three principal stresses at a point. "σ_1" is maximum normal stress that can act on any plane passing through the point "σ_3" is the minimum normal stresses that can act on planes passing through the point

- $\frac{\sigma_1 - \sigma_3}{2}$ is the maximum shear stress that can act on any plane passing through the point

- Von Mises stress

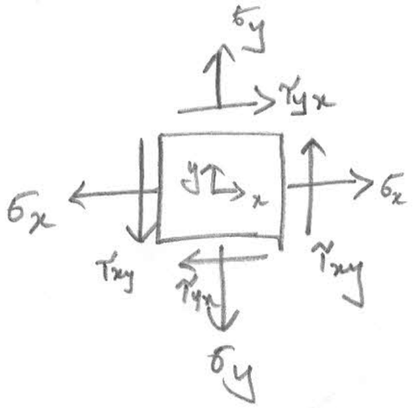
$$\underline{\sigma_{\text{von Mises}}} = \sqrt{\left\{ \frac{1}{2} \left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right] + 3 \left(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2 \right) \right\}}$$

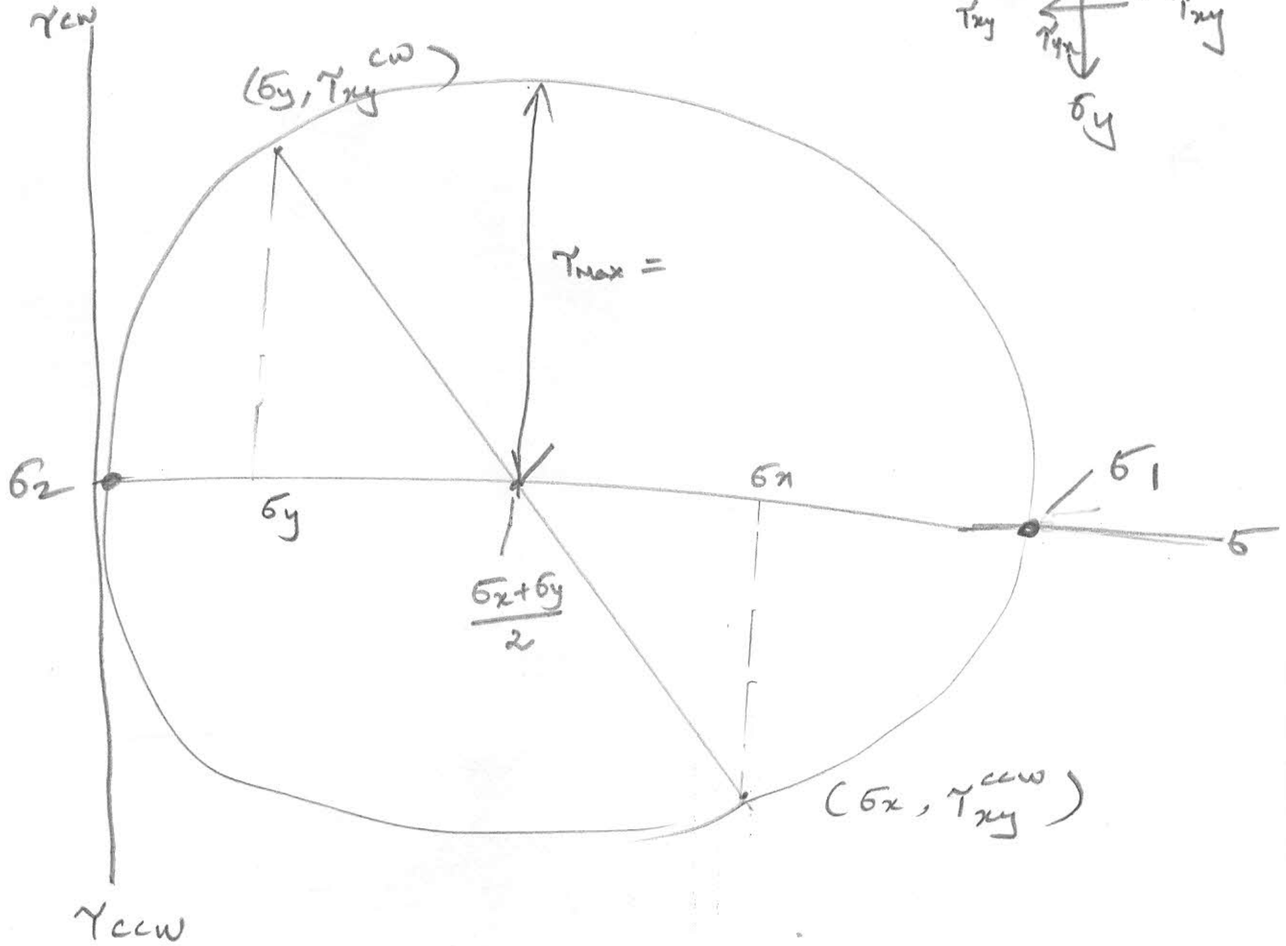
$$\sqrt{\frac{1}{2} \left\{ 3 \, tr[\sigma]^2 - (tr[\sigma])^2 \right\}}$$

Thus $\sigma_{\text{von Mises}}$ is a stress invariant

Mohr circle for 2D state of stress

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$





$$\sigma_1 = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Generalized Hooke's Law for Isotropic, Linear-Elastic Material Behavior

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\,tr[\sigma]\,\delta_{ij}$$

$(i,j = 1,2,3)$

or

$$\boxed{[\epsilon] = \frac{1+\nu}{E}[\sigma] - \frac{\nu}{E}\,tr[\sigma]\,[I]}$$

where

ν is Poisson's ratio

E is Young's modulus

I is identity matrix, i.e. $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thus

$$\epsilon_{11} = \frac{\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}}{E}$$

$$\epsilon_{22} = \frac{\sigma_{22} - \nu\sigma_{33} - \nu\sigma_{11}}{E}$$

$$\epsilon_{33} = \frac{\sigma_{33} - \nu\sigma_{11} - \nu\sigma_{22}}{E}$$

$$\epsilon_{12} = \frac{1+\nu}{E}\sigma_{12} \quad \text{or} \quad \boxed{\begin{aligned}\gamma_{12} &= 2\epsilon_{12} \\ &= G\,\sigma_{12}\end{aligned}}$$

$$\epsilon_{13} = \frac{1+\nu}{E}\sigma_{13} \quad \text{or} \quad \boxed{\gamma_{13} = G\,\sigma_{13}}$$

$$\epsilon_{23} = \frac{1+\nu}{E}\sigma_{23} \quad \text{or} \quad \boxed{\gamma_{23} = G\,\sigma_{23}}$$

Equivalently, it can be shown that

$$\sigma_{ij} = \frac{E}{1+\nu}\epsilon_{ij} + \frac{\nu E}{(1-2\nu)(1+\nu)} \operatorname{tr}[\epsilon]\,\delta_{ij}$$

or

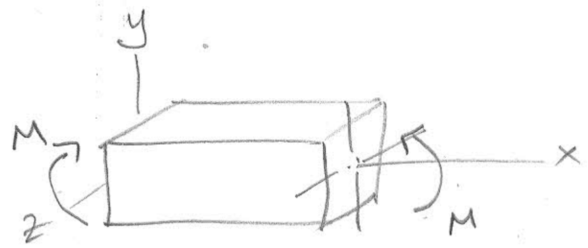
$$\boxed{[\sigma] = \frac{E}{1+\nu}[\epsilon] + \frac{\nu E}{(1-2\nu)(1+\nu)} \operatorname{tr}[\epsilon]\, I}$$

Normal stresses in bending

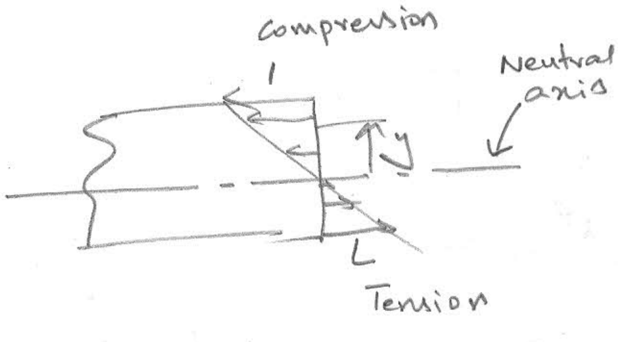
Assumptions

- Beam is subjected to pure bending
- Material is isotropic and homogenous
- Material obeys generalized Hooke's law
- Plane cross-sectoons of beam remain plane during bending (shear strain is zero ; Euler - Bernoulli beam theory is assumed)
- Beam is initially straight, i.e. initial curvature of beam is zero
- Beam has an axis of symmetry in plane of bending

$$\sigma_x = - \frac{My}{I}$$



$\boxed{Z = I/c}$ is called section modulus with $c = y_{max}$



Thus $\boxed{\sigma_{max} = \frac{M}{Z}}$