Duality ~~Table~~ Theory:

Every LP has an associated LP called Dual problem & that given LP is called Primal problem.

Watch maker's problem:

Smith family father (F) & son (S)

Watch type	Profit/unit ($)	Working hrs required F	S	Decision variables
1	60	2	3	x_1
2	40	1	4	x_2
3	80	4	2	x_3

max. working hrs : ~~60 hrs per week.~~ 50 & 60 hrs per week

max. ~~$W = x_1 + x_2 + x_3$~~

$W = 60x_1 + 40x_2 + 80x_3$

st. $2x_1 + x_2 + 4x_3 \leq$ ~~60~~ 50 . . . (I)

$3x_1 + 4x_2 + 2x_3 \leq 60$ . . . (II)

$x_1, x_2, x_3 \geq 0$

John ~~Black~~ Blacke competetor

dealing with same type of business.

Father's wage $- y_1$ \$ per hr.

Son's wage $- y_2$ \$ per hr.

min. $B = 50y_1 + 60y_2$

st. $2y_1 + 3y_2 \geq 60$

$y_1 + 4y_2 \geq 40$

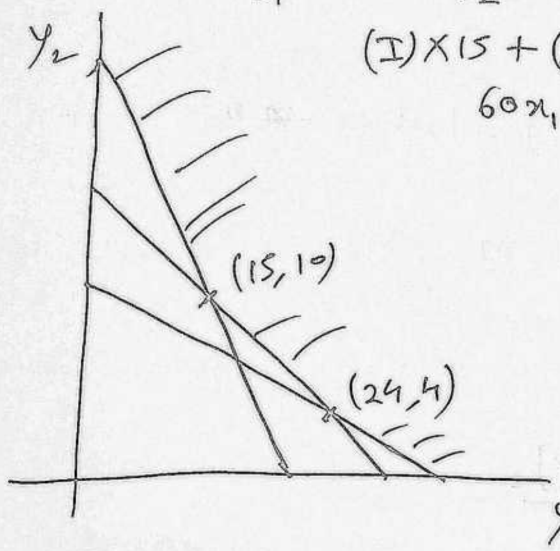
$4y_1 + 2y_2 \geq 80$

$y_1, y_2 \geq 0$

Solⁿ: Solve graphically

$y_1 = 15 \quad y_2 = 10 \quad B = \$1350$ per week

$(I) \times 15 + (II) \times 10$



$$60x_1 + 55x_2 + 80x_3 \le 1350 \le 50y_1 + 60y_2$$

Best value of $W = 60x_1 + 40x_2 + 80x_3 = 1350$ if $x_2 = 0$

if $x_2 = 0$ then solution will lie at the intersection of (I) & (II)

$$2x_1 + 4x_3 \le 50$$
$$3x_1 + 2x_3 \le 60$$

$\Rightarrow x_1 = \frac{3}{4} \quad x_2 = 15/4$

$W = 1350$

Diet Problem

~~n → no. of~~ n → foods

m → neutrients

$$\min.\ z = \sum_{j=1}^{n} c_j x_j$$

$$\text{st.} \quad \sum_{j=1}^{n} a_{ij} x_j \ge b_i \quad \forall i = 1, 2, \ldots m$$

$$x_j \ge 0$$

$y_i \rightarrow$ Price of i th neutrient containing unit neutrient i

max.

$$W = \sum_{i=1}^{m} b_i y_i$$

$$\sum_{j=1}^{m} a_{ij} y_i \leq c_j \quad \forall\, j = 1, 2 \ldots n$$

$$y_i \geq 0 \quad \forall\, i = 1, 2 \ldots, m$$

How to get dual [D] from Primal [P]:

Primal: Product mix problem

Max: $z = c_1 x_1 + c_2 x_2 + \ldots c_n x_n$

st.

m constraints

$$a_{11} x_1 + a_{12} x_2 + \ldots a_{1n} x_n = b_1 \qquad y_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots a_{2n} x_n = b_2 \qquad y_2$$

$$\vdots$$

$$a_{n1} x_1 + a_{n2} x_2 + \ldots a_{nn} x_n = b_n \qquad y_m$$

$$x_i \geq 0 \quad \forall\, i = 1 \ldots n$$

Dual variable: $y_1, y_2, \ldots, y_m$

Dual problem:

min $W = b_1 y_1 + b_2 y_2 + \ldots b_n y_n$

st.

$$a_{11} y_1 + a_{21} y_2 + \ldots a_{m1} y_m \geq c_1$$

$$a_{12} y_1 + a_{22} y_2 + \ldots a_{m2} y_m \geq c_2$$

$$\vdots$$

$$a_{1n} y_1 + a_{2n} y_2 + \ldots a_{mn} y_m \geq c_n$$

$$y_1, y_2 \ldots y_m \geq 0$$

optimal pricing of resources.

Matrix Form:

[P] max. $z = \underline{c}\,\underline{x}$

st. $\underline{A}\,\underline{x} = \underline{b}$

$\underline{x} \geq 0$

[D] min. $W = \underline{y}\,\underline{b}$

st. $\underline{y}\,\underline{A} \geq \underline{c}$ $\quad (\underline{y}_{1\times m},\ \underline{A}_{m\times n},\ \underline{c}_{1\times n})$

$\underline{y} \geq 0$

Rules to convert primal to dual for LP of any form:

Primal ⟷	Dual
Max. type	Min- type
profit coefficient	RHS
RHS	cost coefficient
constraints ($\leq$) ⟷	variables (≥ 0)
($=$)	& unrestricted
($\geq$)	(≤ 0)
Variables	Constraints
(≥ 0)	($\geq$)
unrestricted	($=$)
(≤ 0)	($\leq$)

Example: max: $z = x_1 + 4x_2 + 3x_3$

St. $2x_1 + 3x_2 - 5x_3 \leq 2$ — Dual variables y_1

$3x_1 - x_2 + 6x_3 \geq 1$ — y_2

$x_1 + x_2 + x_3 = 4$ — y_3

$x_1 \geq 0$, $x_2 \leq 0$, x_3 unrestricted.

Dual of above problem

min. $w = 2y_1 + y_2 + 4y_3$

St. $2y_1 + 3y_2 + y_3 \geq 1$

$3y_1 - y_2 + y_3 \leq 4$

$-5y_1 + 6y_2 + y_3 = 3$

$y_1 \geq 0$, $y_2 \leq 0$, y_3 is unrestricted

Min: $z = 2x_1 + x_2 - x_3$

St. $x_1 + x_2 - x_3 = 1$ — y_1

$x_1 - x_2 + x_3 \geq 2$ — y_2

$x_2 + x_3 \leq 4$ — y_3

$x_1 \geq 0$, $x_2 \leq 0$, x_3 unconstraint

Dual max: $W = y_1 + 2y_2 + 4y_3$

St. $y_1 + y_2 \leq 2$

$y_1 - y_2 + y_3 \geq 1$

$-y_1 + y_2 + y_3 = -1$

y_1 unrestricted, $y_2 \geq 0$, $y_3 \leq 0$

Duality Theorem:

Theorem: The dual problem of the dual is the primal

Proof: (D) min $W = \underline{y}\,\underline{b}$ $\longrightarrow$ max. $-W = -\underline{y}\,\underline{b}$ Dual var.

st. $\underline{y}\,\underline{A} \geq \underline{c}$ $-\underline{y}\,\underline{A} \leq -\underline{c}$ $\underline{x}$

$\underline{y} \geq 0$ $\underline{y} \geq 0$

Dual ↙

Min. $-\underline{c}\,\underline{x}$

st. ~~$-\underline{A}\,\underline{x} \leq -\underline{b}$~~ $-\underline{A}\,\underline{x} \geq -\underline{b}$

$\underline{x} \geq 0$

$\Downarrow$

Max. $\underline{c}\,\underline{x}$

St. $\underline{A}\,\underline{x} \leq \underline{b}$

$\underline{x} \geq 0$

Weak duality theorem:

If $\underline{x}$ is a feasible solⁿ to the primal problem
& $\underline{y}$ is a feasible solⁿ to the dual problem

~~& y~~ Then $\underline{c}\,\underline{x} \leq \underline{y}\,\underline{b}$

Proof: if $\underline{x}$ is a feasible solⁿ to the primal

$\Rightarrow \underline{A}\,\underline{x} \leq \underline{b}$ . . . ①

$\underline{x} \geq 0$ . . . . ②

if $\underline{y}$ is a feasible solⁿ to the dual

$\underline{y}\,\underline{A} \geq \underline{c}$ . . . . ③

~~$y \leq$~~ $\underline{y} \geq 0$ . . . ④

$\underline{y} \times ① \Rightarrow \underline{y}\,\underline{A}\,\underline{x} \le \underline{y}\,\underline{b} \quad \ldots ⑤$

$\underline{x} \times ③ \Rightarrow \underline{y}\,\underline{A}\,\underline{x} \ge \underline{c}\,\underline{x} \quad \ldots ⑥$

from ⑤ & ⑥

$$\underline{c}\,\underline{x} \le \underline{y}\,\underline{A}\,\underline{x} \le \underline{y}\,\underline{b}$$

$$\Rightarrow \underline{c}\,\underline{x} \le \underline{y}\,\underline{b}$$

If primal is unbounded i.e. $\underline{c}\,\underline{x} \to \infty$

then dual is infeasible.

If dual is unbounded $\underline{y}\,\underline{b} \to -\infty$ then primal is infeasible

If x^* is a feasible solⁿ to the primal problem

& y^* — " — dual — "

& $\underline{c}\,\underline{x}^* = \underline{y}^*\,\underline{b}$

then x^* & y^* are the optimal solⁿ to the respective problem.

Proof:- from duality theorem

$$\underline{c}\,\underline{x} \le \underline{y}\,\underline{b} \quad \text{for feasible } x \text{ \& } y$$

$$\underline{c}\,\underline{x}^* \le \underline{y}\,\underline{b} \quad \text{for feasible } x^* \text{ \& } y$$

from given conⁿ $\underline{c}\,\underline{x}^* = \underline{y}^*\,\underline{b}$

$$\underline{y}^*\,\underline{b} \le \underline{y}\,\underline{b} \quad \text{for any feasibility}$$

$\Rightarrow y^*$ is optimal to Dual problem.

Strong Duality Theorem:

Proposition: If B is the optimal primal Basis, then optimal sol^n of the dual problem is given by $\underline{y}^* = \underline{C}\,\underline{B}^{-1}$

Proof: $\underline{c}\,\underline{x}^* = \underline{y}^*\,\underline{b}$

$$\Rightarrow \underline{c}_B \underline{B}^{-1} \underline{b} = \underline{y}^* \underline{b} \Rightarrow \underline{y}^* = \underline{c}_B \underline{B}^{-1}$$

Duality Theory:

- Primal to dual
- WDT $\rightarrow \underline{c}\,\underline{x} \leq \underline{y}\,\underline{b}$
- SDT $\rightarrow \underline{c}\,\underline{x} = \underline{y}\,\underline{b}$

$\underline{B} \rightarrow$ optimal basis $\quad \underline{y}^* = \underline{c}_B\,\underline{B}^{-1}$

Economic interpolation of dual variables:

Optimal basis of the primal problem $Z = \underline{c}_B\,\underline{B}^{-1}\,\underline{b}$

Let B be optimal basis of the primal problem

Suppose i th resource is changed from b_i to Δb_i $\quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

& optimal primal basis remains unchanged

$$Z' = \underline{c}_B\,\underline{B}^{-1}\,\underline{b}' \qquad \underline{b}' = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i + \Delta b_i \\ \vdots \\ b_m \end{bmatrix}$$

$$\Delta Z = Z' - Z = c_B B^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \Delta b_i \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} y_1^* & y_2^* & \dots & y_m^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \Delta b_i \\ \vdots \\ 0 \end{bmatrix} \qquad \Delta Z_i = y_i^* \Delta b_i$$

$$y_i^* = \frac{\Delta z_i}{\Delta b_i} \rightarrow \text{shadow price of } i\text{th resource}$$

& dual price

Complimentry Slackness Theorem (CST)

Primal (P) max $\underline{c}\,\underline{x}$, st. $\underline{A}\,\underline{x} < \underline{b}$ $\underline{x} \geq 0$

Dual (D) min $\underline{y}\,\underline{b}$, st. $\underline{y}\,\underline{A} \geq \underline{c}$, $\underline{y} \geq 0$

If $\underline{x}$ & $\underline{y}$ are feasible solⁿ to the primal (P) & dual (D), then $\underline{x}$ & $\underline{y}$ are optimal to their respective problem

$$(\underline{y}\,\underline{A} - \underline{c})\,\underline{x} + \underline{y}\,(\underline{b} - \underline{A}\,\underline{x}) = 0$$

Proof: ~~For~~ From Primal

$$\underline{A}\,\underline{x} + \underline{u} = \underline{b} \quad \ldots (1) \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

From Dual

$$\underline{y}\,\underline{A} - \underline{v} = \underline{c} \quad \ldots (2) \qquad v = [v_1 \; v_2 \; \cdots \; v_n]$$

$$\underline{y} \times (1) \Rightarrow \underline{y}\,\underline{A}\,\underline{x} + \underline{y}\,\underline{u} = \underline{y}\,\underline{b} \quad \ldots (3)$$

$$(2) \times \underline{x} \Rightarrow \underline{y}\,\underline{A}\,\underline{x} - \underline{v}\,\underline{x} = \underline{c}\,\underline{x} \quad \ldots (4)$$

$$(3) - (4) \Rightarrow \underline{y}\,\underline{u} + \underline{v}\,\underline{x} = \underline{y}\,\underline{b} - \underline{c}\,\underline{x} \quad \ldots (5)$$

If $\underline{x}$ & $\underline{y}$ are optimal $\Rightarrow \underline{c}\,\underline{x} = \underline{y}\,\underline{b}$

from (5) we have $\underline{v}\,\underline{x} + \underline{y}\,\underline{u} = 0$

on the other hand if $\underline{y}\,\underline{u} - \underline{v}\,\underline{x} = 0$

from (5) we have $\underline{y}\,\underline{b} = \underline{c}\,\underline{x}$ from SDT

Implications of CST, CST conditions

since, $u, v, x, y \geq 0$

$\Rightarrow \underline{y}\,\underline{u} = \underline{v}\,\underline{x} = 0$

from $\underline{y}\,\underline{u} = 0 \Rightarrow [y_1, y_2 \ldots y_m] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = 0$

$\Rightarrow y_i u_i = 0 \quad \forall i = 1, 2 \ldots m$

if $y_i > 0 \Rightarrow u_i = 0$

if ~~$u_i \geq$~~ $u_i > 0 \Rightarrow y_i = 0$

Primal: Max $z = \underline{c}\,\underline{x}$ — slack var. $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$

st. $\underline{A}\,\underline{x} \leq \underline{b}$

$\underline{x} \geq 0$

Dual: min: $\underline{w} = \underline{y}\,\underline{b}$ — surplus var. $v = [v_1 \; v_2 \ldots v_n]$

$\underline{y}\,\underline{A} \geq \underline{c}$

$\underline{y} \geq 0$

$y_i u_i = 0 \; ; \; v_j x_j = 0$

Ex: Min: $W = 2y_1 + 3y_2 + 5y_3 + 2y_4 + 3y_5$

s.t. $y_1 + y_2 + 2y_3 + y_4 + y_4 + 3y_5 \geq 4$

$2y_1 - 2y_2 + 3y_3 + y_4 + y_5 \geq 3$

$y_i \geq 0$

dual variable	sur. var.
x_1	v_1
x_2	v_2

dual problem: Max. $z = 4x_1 + 3x_2$

slack var.	st.
$u_1 = 0$	$x_1 + 2x_2 \leq 2$
$u_2 > 0$	$x_2 - 2x_2 \leq 3$
$u_3 > 0$	$2x_1 + 3x_2 \leq 5$
$u_4 > 0$	$x_1 + x_2 \leq 2$
$u_5 = 0$	$3x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$

Solve using graphical method.

$x_1 = 4/5$, $x_2 = 3/5$, $Z = 5$

From CST con^n.

$x_1 > 0 \Rightarrow v_1 = 0$

$x_2 > 0 \Rightarrow v_2 = 0$

$u_2 > 0 \Rightarrow y_2 = 0$ $u_3 > 0 \Rightarrow y_3 = 0$ $u_4 > 0 \Rightarrow y_4 = 0$

From primal constraints

$y_1 + 3y_5 = 4$; $2y_1 + y_5 = 3$ } $y_1 = 1$, $y_5 = 1$ $W = 5$

Dual Simplex method:

Suppose $\underline{B}$ is a feasible primal basis.

$\Rightarrow \underline{x}_B = \underline{B}^{-1}\underline{b} \geq 0$

Let this $\underline{B}$ is optimal basis for the optimal problem

$\Rightarrow \underline{c}_B \underline{B}^{-1}\underline{A} - \underline{c} \geq 0 \Rightarrow \underline{y}\,\underline{A} - \underline{c} \geq 0$ $\underline{c}_B \underline{B}^{-1} \geq 0 \Rightarrow \underline{y} \geq 0$

Primal simplex:

Initial $\underline{B}$ → Feasible for primal
infeasible for dual

$\underline{B}'$ → —ıı— primal
—ıı— dual

optimal $\underline{B}^*$ → Feasible for primal
—ıı— for dual.

From: $\underline{c}\,\underline{x} = \underline{y}\,\underline{b}$

Algorithm:

① Problem must be max. type with ≤ type constraint.

② updated profit coefficient ≥ 0 ⇒ dual feasible

③ Atleast one RHS is -ve. ⇒ principal infeasibility

④ Leaving variable ⇒ most -ve RHS.

⑤ Entering variable Max. ratio rule
if x_k is leaving variable
find ratio for which $\dfrac{z_j - c_j}{a_{jk}}$, $a_{jk} < 0$

Ex. min. $z = 3x_1 + 2x_2$

st. $3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$

Convert into max. type with $\leq$ type constraints.

max. $z' = -3x_1 - 2x_2$ — Slack variables.

st. $-3x_1 - x_2 \leq -3$ — x_3

$-4x_1 - 3x_2 \leq -6$ — x_4

$x_1 + x_2 \leq 3$ — x_5

$x_1, x_2 \geq 0$

$\Rightarrow$

$-3x_1 - x_2 + x_3 = -3$

$-4x_1 - 3x_2 + x_4 = -6$

$x_1 + x_2 + x_5 = 3$

$x_i \geq 0$

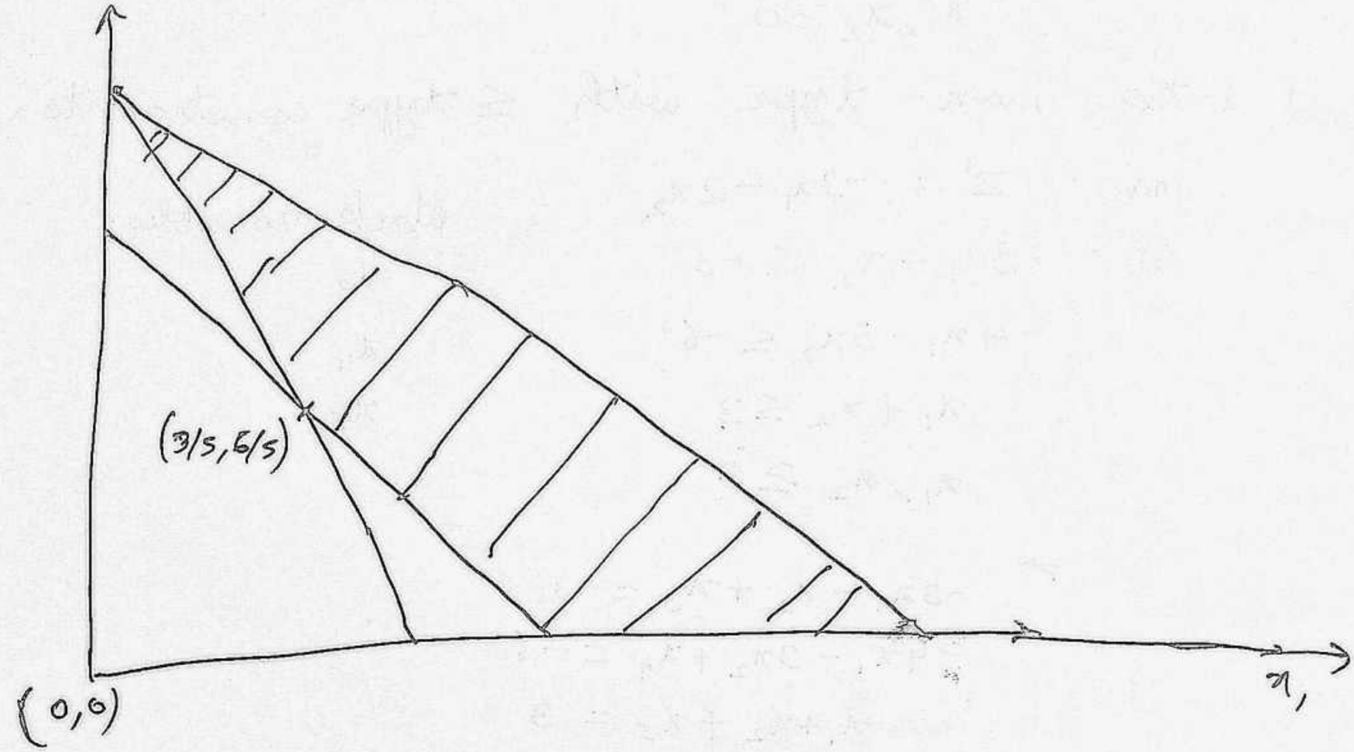
Basis	x_1	x_2 ↓	x_3	x_4	x_5	RHS
x_3	-3	-1	1	0	0	-3
x_4	-4	(-3)	0	1	0	-6 → leaving var.
x_5	1	1	0	0	1	3
z'	3	2	0	0	0	0
Ratio	3/(-4) ↓	2/(-3)	—	—	—	
x_3	(-5/3)	0	1	-1/3	0	-1 → leav.
x_2	4/3	1	0	-1/3	0	2
x_5	-1/3	0	0	1/3	1	1
$z_j - c_j$ (z')	1/3	0	0	2/3	0	-4
Ratio	$\frac{1/3}{-5/3} = -\frac{1}{5}$	—	—	(2/3)/(-1/3) = -2	—	
x_1	1	0	-3/5	1/5	0	3/5
x_2	0	1	-4/5	-3/5	0	6/5
x_3	0	0	-1/5	2/5	1	6/5
z'	0	0	1/5	3/5	0	-21/5

$$B^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

optimal solⁿ:

$x_1 = 3/5 \quad x_2 = 6/5 \quad z = -z' = 21/5$



sensitivity Analysis: (for any number of variables)

Parameters of LP a_{ij}, b_i, c_j

Simplex table in matrix form:

Basis	$\underline{x}^T$	$\underline{x}_j^T$	RHS
$\underline{x}_B$	$\underline{B}^{-1}\underline{A}$	$\underline{B}^{-1}$	$\underline{B}^{-1}\underline{b}$
$z_j - c_j$	$\underline{c}_B \underline{B}^{-1}\underline{A} - \underline{c}$	$c_B B^{-1}$	$\underline{c}_B \underline{B}^{-1}\underline{b}$

$$z_j - c_j = c_B B^{-1} a_j - c_j$$

$$\underline{A} = [a_1 \ a_2 \ \ldots \ a_n]$$

Example: (Products A_1, B_1, C_1 → x_1, x_2, x_3)

Max: $z = 2x_1 + 3x_2 + x_3$

s.t. $\frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \le 1$ (labour)

$\frac{1}{3}x_1 + \frac{4}{3}x_2 + \frac{7}{3}x_3 \le 3$ (Material)

$x_1, x_2, x_3 \ge 0$

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_1	1	0	4	4	-1	1
x_2	0	1	2	-1	1	2
$z_j - c_j$	0	0	3	5	1	8

Change in c_j

(1) Change in c_j of a non basic variable

Find the range of c_3 for which the current basis remains optimal (feasible)

$z_j - c_j = c_B B^{-1} a_j - c_j \qquad A = [a_1 \; a_2 \; \cdots \; a_n]$

$$z_3 - c_3 = c_B B^{-1} a_3 - c_3 = \begin{bmatrix} 5 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 7/3 \end{bmatrix} - c_3$$

$$= 4 - c_3$$

The basis (solⁿ) remains optimal if $z_3 - c_3 \ge 0$

$\Rightarrow 4 - c_3 \ge 0$

$c_3 \le 4$

$$A = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 4/3 & 2/3 \end{bmatrix}$$

if c_3 is changed from 1 to 6

$$z_3 - c_3 = 4 - 6 = -2$$

BV	x_1	x_2	x_3 ↓	x_4	x_5	RHS	Ratio
x_1	1	0	-1	4	-1	1	—
x_2	0	1	2	-1	1	2	2/2 = 1
$z_j - c_j$	0	0	-2	5	1	8	

Apply primary simplex to find new optimal solⁿ

(2) Change in c_j of a basic variable

Find the range of c_j for which the current basis remains optimal (feasible)

only all $z_j - c_j$ $\forall i = 1, 2 \ldots n$

$$c_B = [c_1 \quad 3]$$

$$z_1 - c_1 = c_B B^{-1} a_1 - c_1 = [c_1 \ 3]\begin{bmatrix}1\\0\end{bmatrix} - c_1 = 0$$

$$z_2 - c_2 = c_B B^{-1} a_2 - c_2 = [c_1 \ 3]\begin{bmatrix}0\\1\end{bmatrix} - 3 = 0$$

→ They are coeff. of basic variables.

$$z_3 - c_3 = c_B B^{-1} a_3 - c_3 = [c_1 \ 3]\begin{bmatrix}-1\\2\end{bmatrix} - 1 = -c_1 + 5$$

~~$c_B B^{-1} = [c_1 \ 3]\begin{bmatrix}4\\-1\end{bmatrix}$~~ $\quad c_B B^{-1} = [c_1 \ 3]\begin{bmatrix}4 & -1\\-1 & 1\end{bmatrix} = [4c_1 - 3 \quad c_1 + 3]$

$$z_4 - c_4 = 4c_1 - 3$$
$$z_5 - c_5 = -c_1 + 3$$

for optimality of the basic feasible solution $z_j - c_j \geq 0 \ \forall j$

$z_3 - c_3 = -c_1 + 5 \Rightarrow c_1 \leq 5$

$z_4 - c_4 \geq 0 \Rightarrow 4c_1 - 3 \geq 0 \Rightarrow c_1 \geq \frac{3}{4}$

$z_5 - c_5 \geq 0 \Rightarrow -c_1 + 3 \geq 0 \Rightarrow c_1 \leq 3$

$$\frac{3}{4} \leq c_1 \leq 3$$

(3) Change c_j of both basic & non-basic variables

New obj. $z = x_1 + 4x_2 + 2x_3$

$c' = [1 \ 4 \ 1]$

all c_1, c_2, c_3 are within the optimality range

& $z_1 - c_1 = 0$; $z_2 - c_2 = 0$ $z_3 - c_3 = 6$ $z_4 - c_4 = 0$ → multiple optimal solution.

$z_5 - c_5 = 3$

Sensitivity analysis:

- change in c_j
  - c_j of basic variable
  - c_j of non basic variable
  - c_j of BV & NBV

BV	x^T	x_s^*	RHS
x_B	$B^{-1}A$	B^{-1}	$B^{-1}b$
$z_j - c_j$	$C_B B^{-1}A - C$	$C_B B^{-1}$	$C_B B^{-1} b$

& Max. $z = 2x_1 + 3x_2 + x_3$

St. $\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3} \leq 1$ (labour)

$\frac{x_1}{3} + \frac{4x_2}{3} + \frac{7x_3}{3} \leq 3$ (material)

Simplex table:

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_1	1	0	-1	4	-1	1
x_2	0	1	2	-1	1	2
$z_j - c_j$	0	0	3	5	1	8

$b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

<u>Change in RHS</u>:

Find the range of b_1 for which the current basis remains feasible and optimal.

$$x_B = B^{-1} b = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4b_1 - 3 \\ -b_1 + 3 \end{bmatrix}$$

For the current basis remains feasible

$x_B \geq 0 \Rightarrow$ $4b_1 - 3 \geq 0$; $b_1 \geq 3/4$

$-b_1 + 3 \geq 0$; $b_1 \leq 3$

$$\frac{3}{4} \leq b_1 \leq 3$$

$$x_B = B^{-1} b = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \not{5} \\ b_2 \end{bmatrix} = \begin{bmatrix} \not{20}\ 4 - b_2 \\ -\not{5} + b_2 \end{bmatrix}$$

$4\ \not{20} - b_2 \geq 0$, $b_2 \leq \not{20}\ 4$

$-\not{5} + b_2 \geq 0$ $b_2 \geq \not{5}\ 1$

~~$b_2 \in [5, 20]$~~ $b_2 \in [1, 4]$

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_1	1	0	-1	4	-1	13
x_2	0	1	2	-1	1	-1
$z_j - c_j$	0	0	3	5	1	23

$$[A \,;\, I] = \begin{bmatrix} 1/3 & 1/3 & 1/3 & \vdots & 1 & 0 \\ 1/3 & 4/3 & 7/3 & \vdots & 0 & 1 \end{bmatrix} \quad (x_1, x_2, x_3, x_4, x_5)$$

$$B = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 4/3 \end{bmatrix} \quad (x_1, x_2)$$

The current sol^n (13, −1) is optimal but not feasible for the primal. So we apply dual simplex to find the optimal for dual

BV	x_1	x_2	x_3	x_4 ↓	x_5	RHS
x_1	1	0	−1	4	−1	13
x_2	0	1	2	(−1)	1	−1
$z_j - c_j$	0	0	3	5	1	13
Ratio	–	–	–	5/(−1)	–	
x_1	1	4	8	0	3	9
x_4	0	−1	−2	1	−1	1
$z_j - c_j$	0	5	13	0	6	8

<u>Change in A:</u>

) Adding a new variable (Introducing a new product 'D')

product 'D' : Profit coefficient = 3

requirement: 1 unit of labour, 1 unit of material $\Big\}$ $a_6 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $x_6 \rightarrow$ production quantity of product D

$$B^{-1}a_6 = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$c_B B^{-1} a_6 - c_6 = [5 \;\; 1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3 = 3$$

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_1	Same					Same
x_2						

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_1	Same					3	Same
x_2						0	
	Same					3	Same

BV	x_1	x_2	x_3	x_4	x_5	x_6 ↓	RHS	
x_1	1	0	-1	4	-1	3	1	1/3
x_2	0	1	2	-1	1	0	2	
$z_j - c_j$	0	0	3	5	1	-3	8	
x_6	1/3	0	-1/3	4/3	-1/3	1		
x_2		1				0		

(ii) Variation in value of a_{ij} (change in resource requirement of 'j')

a_{ij} is corresponding to a basic variable is changed

⇒ B^{-1} will change, leads to change in entire simplex table

If the resulting table has neither feasible nor optimal basis then solve the problem from scratch.

a_{ij} corresponding to a non-basic variable is changed.

Eg. a_{13} & a_{23} then only find $B^{-1}a_3$ & $C_B B^{-1} a_3 - c_3$

If $C_B B^{-1} a_3 - c_3$ becomes -ve then apply primal problem

(iii) Adding a new constraint

suppose carbon emission constraint

$$x_1 + 2x_2 + x_3 \leq 10$$

Check whether the new constraint is satisfied

$$1 + 2 \times 2 + 0 = 5 < 10$$

if $x_1 + 2x_2 + x_3 \leq 4 \Rightarrow$ active constraint

in augmented form

$$x_1 + 2x_2 + x_3 + x_6 = 4$$

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_1	1	0	-1	4	-1	0	1
x_2	0	1	2	-1	1	0	2
x_6	1	2	1	0	0	1	4
$z_j - c_j$	0	0	3	5	1	0	8
x_1	1	0	-1	4	-1	0	1
x_2	0	1	2	-1	1	0	2
x_3	0	0	-2	-2	(-1)	1	1
$z_j - c_j$	0	0	3	5	1	0	8
ratio			-3/2	-5/2	-1		
x_1	1	0	1	6	0	-1	2
x_2	0	1	0	-3	0	1	1
x_5	0	0	2	2	1	-1	1
$z_j - c_j$	0	0	1	3	0	1	7