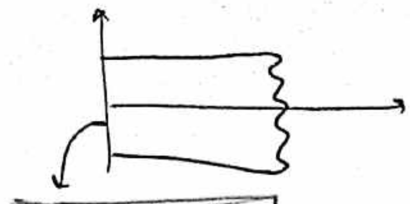
Stokes 2nd Problem



$\boxed{T = T_i + T_1\cos\omega t}$

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}$$

$x=0,\ T = T_i + T_1\cos\omega t$

$x\to\infty,\ T\to T_i$

$t=0,\ T = T_i$

Let $u(x,t) = T - T_i$

$\frac{\partial u}{\partial t} = \frac{\partial T}{\partial t}$ | $\frac{\partial u}{\partial x} = \frac{\partial T}{\partial x}$, $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 T}{\partial x^2}$

$\therefore \boxed{\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}}$ → $x=0,\ u = T_1\cos\omega t$

$x\to\infty,\ u = 0$

$t=0,\ u=0$

(1) steady periodic soln (large time soln)

Let $\boxed{u_c = u_R + iu_I}$ → $u_c(0,t) = T_1\cos\omega t + iT_1\sin\omega t$

$u_c(\infty,t) \to 0$

u_R	u_I
$x=0,\ u_R = T_1\cos\omega t$	$x=0,\ u_I = T_1\sin\omega t$
$x\to\infty,\ u_R\to 0$	$x\to\infty,\ u_I\to 0$

$\therefore u_c = F(x)e^{i\omega t}$

$x=0,\ u_c = F(0)e^{i\omega t} = T_1e^{i\omega t} \Rightarrow \boxed{F(0) = T_1}$

$x\to\infty,\ u_c\to 0 \Rightarrow \boxed{\text{as } x\to\infty,\ F(x)\to 0}$

$\frac{\partial u_c}{\partial t} = F(x)e^{i\omega t}.i\omega$ $\quad \frac{\partial u_c}{\partial x} = F'(x)e^{i\omega t}$

$\frac{\partial^2 u_c}{\partial x^2} = F''(x)e^{i\omega t}$

$\mathcal{L}(u_c) = 0$

$\Rightarrow F e^{i\omega t}.i\omega - \alpha F'' e^{i\omega t} = 0$

$\Rightarrow (Fi\omega - \alpha F'')e^{i\omega t} = 0$

$\Rightarrow Fi\omega - \alpha F'' = 0$

$\Rightarrow Fi\omega = \alpha\frac{d^2F}{dx^2}$ $\quad F = e^{mx}$

$\Rightarrow \cancel{e^{mx}}.i\omega = \alpha m^2\cancel{e^{mx}}$

$\Rightarrow m^2 = \frac{i\omega}{\alpha}$ $\quad\Rightarrow m = \sqrt{\frac{\omega}{\alpha}}\sqrt{i}$

$\Rightarrow \boxed{m = \pm\sqrt{\frac{\omega}{\alpha}}\left(\frac{1+i}{\sqrt{2}}\right)} = \pm\left(\sqrt{\frac{\omega}{2\alpha}} + i\sqrt{\frac{\omega}{2\alpha}}\right)$

$\therefore F(x) = A_1e^{m_1x} + A_2e^{m_2x}$

$F(x) = A_1e^{-\sqrt{\frac{\omega}{2\alpha}}x}.e^{-i\sqrt{\frac{\omega}{2\alpha}}x} + A_2e^{\sqrt{\frac{\omega}{2\alpha}}x}e^{i\sqrt{\frac{\omega}{2\alpha}}x}$

as $x\to\infty$, $F(x)\to 0$ $\therefore A_2 = 0$

$\boxed{\therefore F(x) = A_1e^{-\sqrt{\frac{\omega}{2\alpha}}x}.e^{-i\sqrt{\frac{\omega}{2\alpha}}x}}$

$x=0,\ F = T_1 \quad \therefore A_1 = T_1$

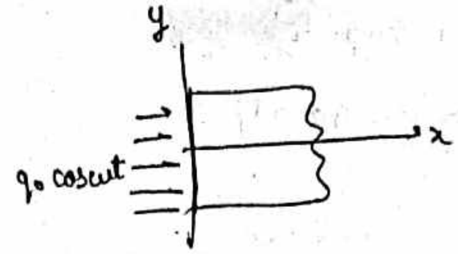
$\therefore u_c = T_1e^{-\sqrt{\frac{\omega}{2\alpha}}x}.e^{-i\sqrt{\frac{\omega}{2\alpha}}x}e^{i\omega t}$

$T = u + T_i$

$\Rightarrow T = T_i + \text{Real}(u_c)$

$\Rightarrow \boxed{T = T_i + T_1e^{-\sqrt{\frac{\omega}{2\alpha}}x}\cos\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}x\right)}$

$\boxed{q = -k\frac{\partial T}{\partial x}}$



$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$x = 0, \quad -k\frac{\partial T}{\partial x} = q_0 \cos\omega t$

$x \to \infty, \quad T \to T_0$

$t = 0, \quad T = T_0$

a $T = T_0 + u$

$\Rightarrow u(x,t) = T(x,t) - T_0$

$\frac{\partial u}{\partial t} = \frac{\partial T}{\partial t}$ | $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 T}{\partial x^2}$ $\qquad \frac{\partial T}{\partial x} = \frac{\partial u}{\partial x}$

$\therefore \boxed{\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}}$ — $x = 0, \quad -k\frac{\partial u}{\partial x} = q_0 \cos\omega t$

$x \to \infty, \quad T = T_0$

$t = 0, \quad T = T_0$

(i) Steady periodic soln

$u_c = u_R + i\,u_I \qquad u_c = F(x)e^{i\omega t}$

$\frac{\partial u_c}{\partial x} = \frac{\partial u_R}{\partial x} + i\frac{\partial u_I}{\partial x}$

$\left.\frac{\partial u_c}{\partial x}\right|_{x=0} = F'(x)e^{i\omega t} = -\frac{q_0}{k}e^{i\omega t}$

$\Rightarrow \boxed{F'(0) = -\frac{q_0}{k}}$

$\boxed{\text{as } x \to \infty, \ F(x) \to 0}$

$\frac{\partial u_c}{\partial t} = F(x)e^{i\omega t}\cdot i\omega \qquad \frac{\partial^2 u_c}{\partial x^2} = F''(x)e^{i\omega t}$

$L(u_c) = 0$

$\Rightarrow (Fi\omega - \alpha F'')e^{i\omega t} = 0$

$\Rightarrow \boxed{m = \pm\left(\sqrt{\frac{\omega}{2\alpha}} + i\sqrt{\frac{\omega}{2\alpha}}\right)}$

$F(x) = A_1 e^{-\sqrt{\frac{\omega}{2\alpha}}x} e^{-i\sqrt{\frac{\omega}{2\alpha}}x} + \overset{0}{\cancel{A_2}}\, e^{+\sqrt{\frac{\omega}{2\alpha}}x} e^{i\sqrt{\frac{\omega}{2\alpha}}x}$

$\boxed{F(x) = A_1 e^{-\sqrt{\frac{\omega}{2\alpha}}x} e^{-i\sqrt{\frac{\omega}{2\alpha}}x}}$

$F'(x) = -\left(A_1 e^{-\sqrt{\frac{\omega}{2\alpha}}x} e^{-i\sqrt{\frac{\omega}{2\alpha}}x}\right)\left(\sqrt{\frac{\omega}{2\alpha}} + i\sqrt{\frac{\omega}{2\alpha}}\right)$

$F'(0) = -\frac{q_0}{k}$

$\Rightarrow \cancel{-}\frac{q_0}{k} = \cancel{-}A_1(1+i)\sqrt{\frac{\omega}{2\alpha}}$

$\Rightarrow A_1 = \frac{q_0}{k}\sqrt{\frac{2\alpha}{\omega}}\frac{1}{(1+i)}$

$= \frac{q_0}{k}\sqrt{\frac{2\alpha}{\omega}}\left(\frac{1-i}{2}\right)$

$\boxed{A_1 = \frac{q_0}{k}\sqrt{\frac{\alpha}{2\omega}}(1-i)}$

$e^{i\pi/2}$