2-D problems

$$-\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)+a_0 u-f=0$$

For Poisson's eqⁿ $\quad a_{12}=a_{21}=a_0=0$

For Laplace eqⁿ $\quad a_{12}=a_{21}=a_2=f=0$

$$\int_\Omega w\left[-\frac{\partial}{\partial x}\underbrace{\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)}_{F_1}-\frac{\partial}{\partial y}\underbrace{\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)}_{F_2}+a_0u-f\right]dx\,dy=0$$

$$\int_\Omega w\left[-\frac{\partial}{\partial x}F_1-\frac{\partial}{\partial y}F_2\right]dx\,dy+\underbrace{\int_\Omega w a_0 u\,dx\,dy}_{A}-\underbrace{\int_\Omega wf\,dx\,dy}_{B}=0$$

$$w\frac{d}{dx}k=\frac{d(wk)}{dx}-k\frac{dw}{dx}$$

$$-w\frac{dk}{dx}=k\frac{dw}{dx}-\frac{d(wk)}{dx}$$

$$\int_\Omega\left(\frac{\partial w}{\partial x}F_1+\frac{\partial w}{\partial y}F_2\right)dx\,dy-\int_\Omega\frac{\partial}{\partial x}(wF_1)\,dx\,dy-\int_\Omega\frac{\partial}{\partial y}(wF_2)\,dx\,dy$$

$$+A-B=0$$

$$\int_\Omega\frac{\partial(wF_1)}{\partial x}dx\,dy=\int_S wF_1 n_x\,ds$$

$$\int_\Omega\frac{\partial(wF_2)}{\partial y}dx\,dy=\int_S wF_2 n_y\,ds$$

$$\int_\Omega\left[\frac{\partial w}{\partial x}\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)+\frac{\partial w}{\partial y}\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)\right]dx\,dy$$

$$+\int_\Omega w a_0 u\,dx\,dy=\int_\Omega wf\,dx\,dy+\int_S w\left(a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}\right)n_x\,ds$$

$$+\int_S w\left(a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}\right)n_y\,ds$$

Simplified form, $\int_\Omega\left(\frac{\partial w}{\partial x}a\frac{\partial u}{\partial x}+\frac{\partial w}{\partial y}a\frac{\partial u}{\partial y}\right)dx\,dy=\int_S wf\,dx\,dy+\int_S wq_n\,ds$

$$q_n = a\left(\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y\right)$$

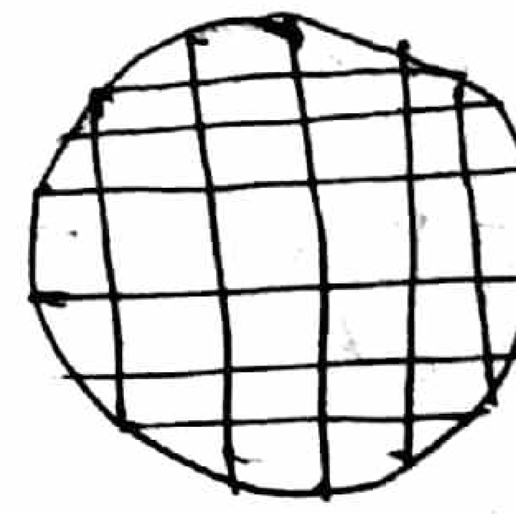
R-R-F $\Rightarrow$ $$\iint_{\Omega} \left[\frac{\partial \psi_i}{\partial x}\, a \sum_{j=1}^{n} \frac{\partial}{\partial x}(u_j \psi_j) + \frac{\partial \psi_i}{\partial y}\, a \sum_{j=1}^{n} \frac{\partial}{\partial y}(u_j \psi_j)\right] dx\,dy$$

ith eqn

$$= \int_{\Omega} \psi_i f \, dx\, dy + \int_{S} \psi_i q_n \, ds$$

Matrix of $n \times n$

The elements are of two types ① Rectangular ② Triangular



1

$x \quad y$

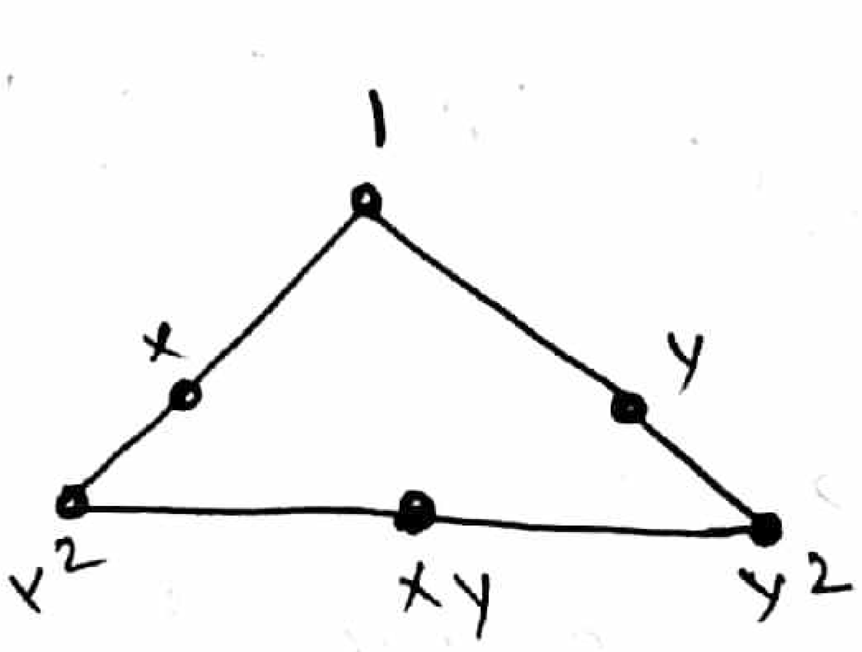
$x^2 \quad xy \quad y^2$

$x^3 \quad x^2y \quad xy^2 \quad y^3$

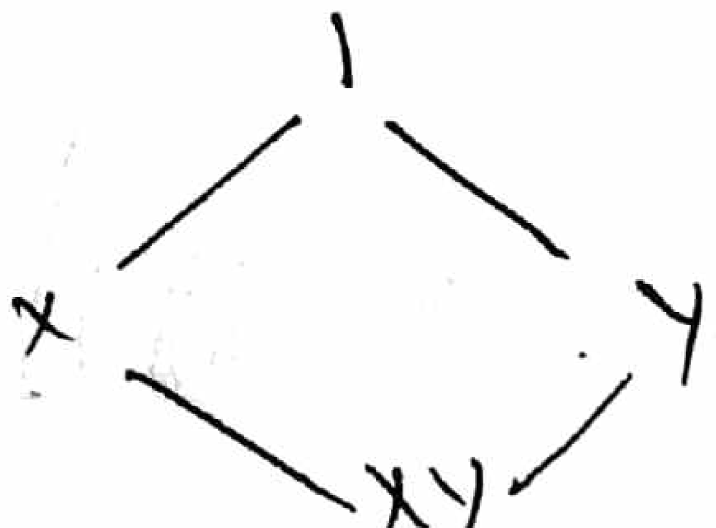
$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$

For quadratic triangular element.

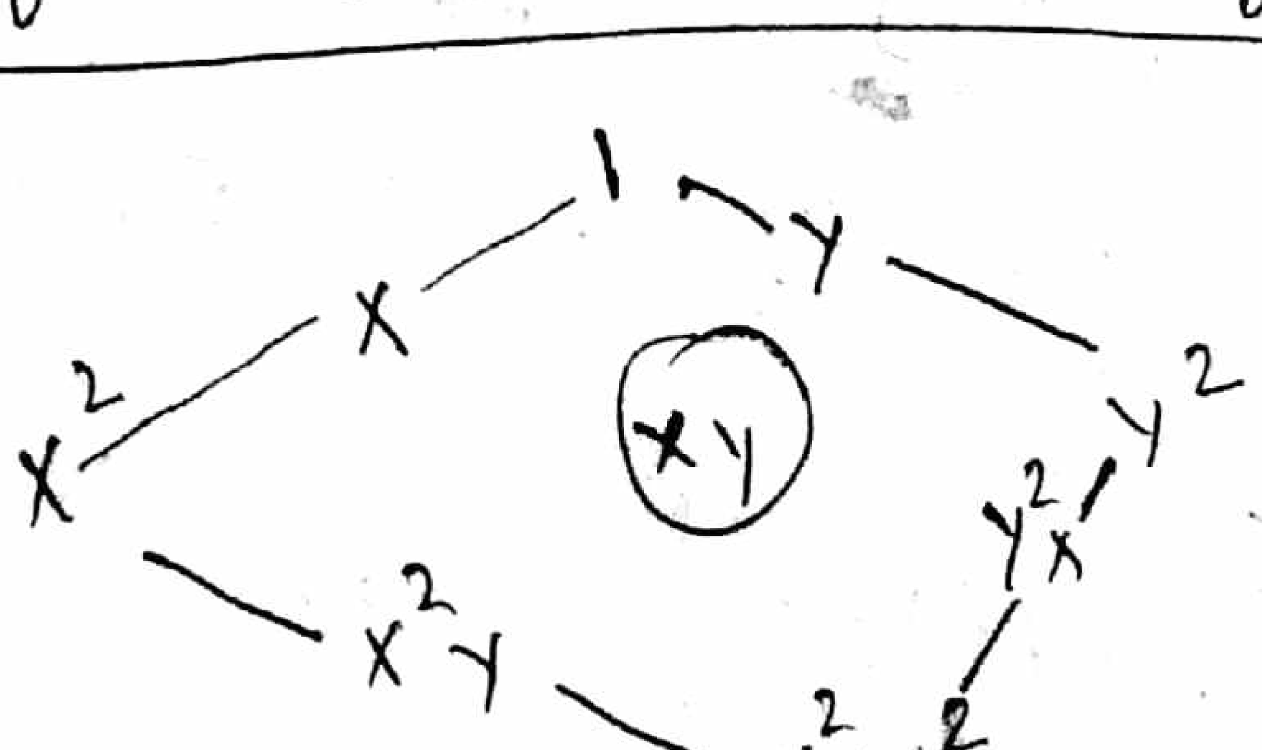
$1 + x + x^2 + xy + y^2 + y$



For Linear rectangular element.



For quadratic ~~linear~~ rectangular element



Total no. of nodal points $= 9$

xy is at the center of the rectangle.

Boundary terms → coefficient of weight functions } for

EBC → ~~coefficient~~ The form of w in weak form } 1-D

In global coordinates { linear triangular element }.

$$\psi_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$

A = area of the triangle.

$$\boxed{\alpha_1 + \alpha_2 + \alpha_3 = 2A}$$

i j k: 1 → 2 → 3

2 → 3 → 1 (i j k)

3 → 1 → 2 (i j k).

$$\alpha_i = x_j y_k - x_k y_j$$

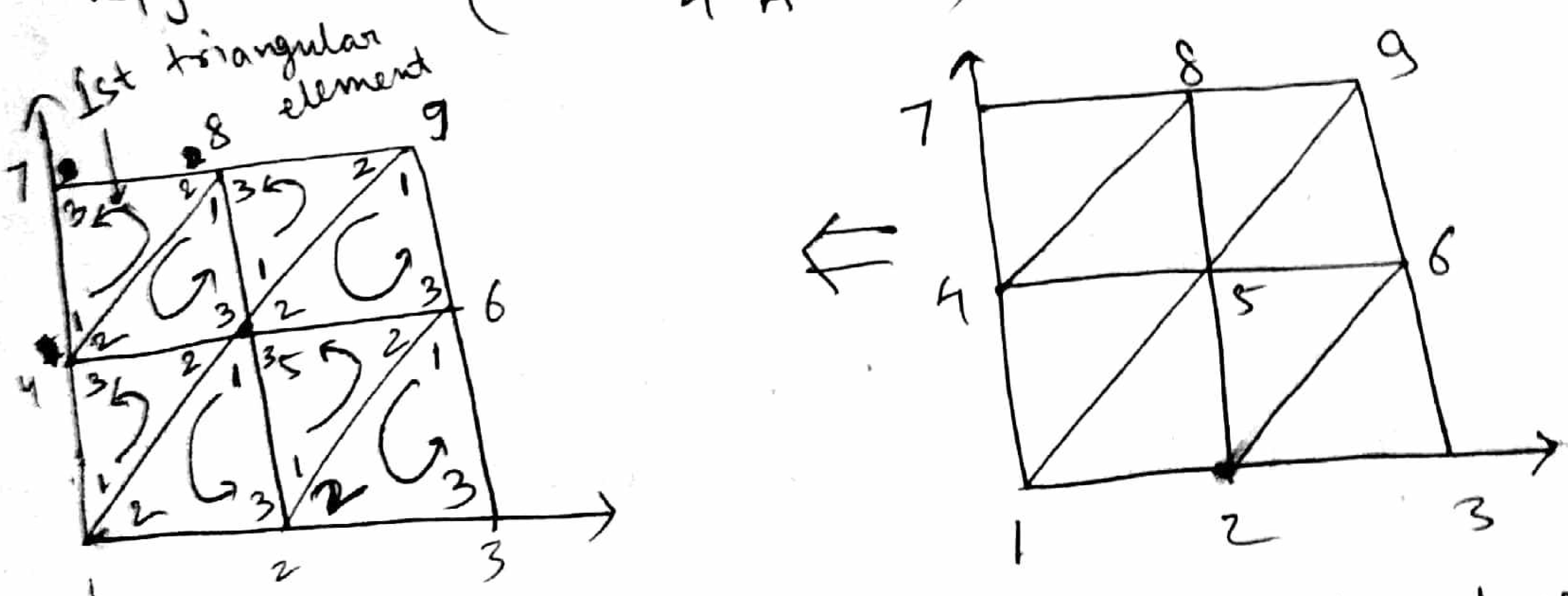
$$\beta_i = y_j - y_k$$

~~$\gamma_i = -(x_j - \ldots)$~~ $\gamma_i = -(x_j - x_k)$

$$\alpha_1 + \alpha_2 + \alpha_3 = 2A$$

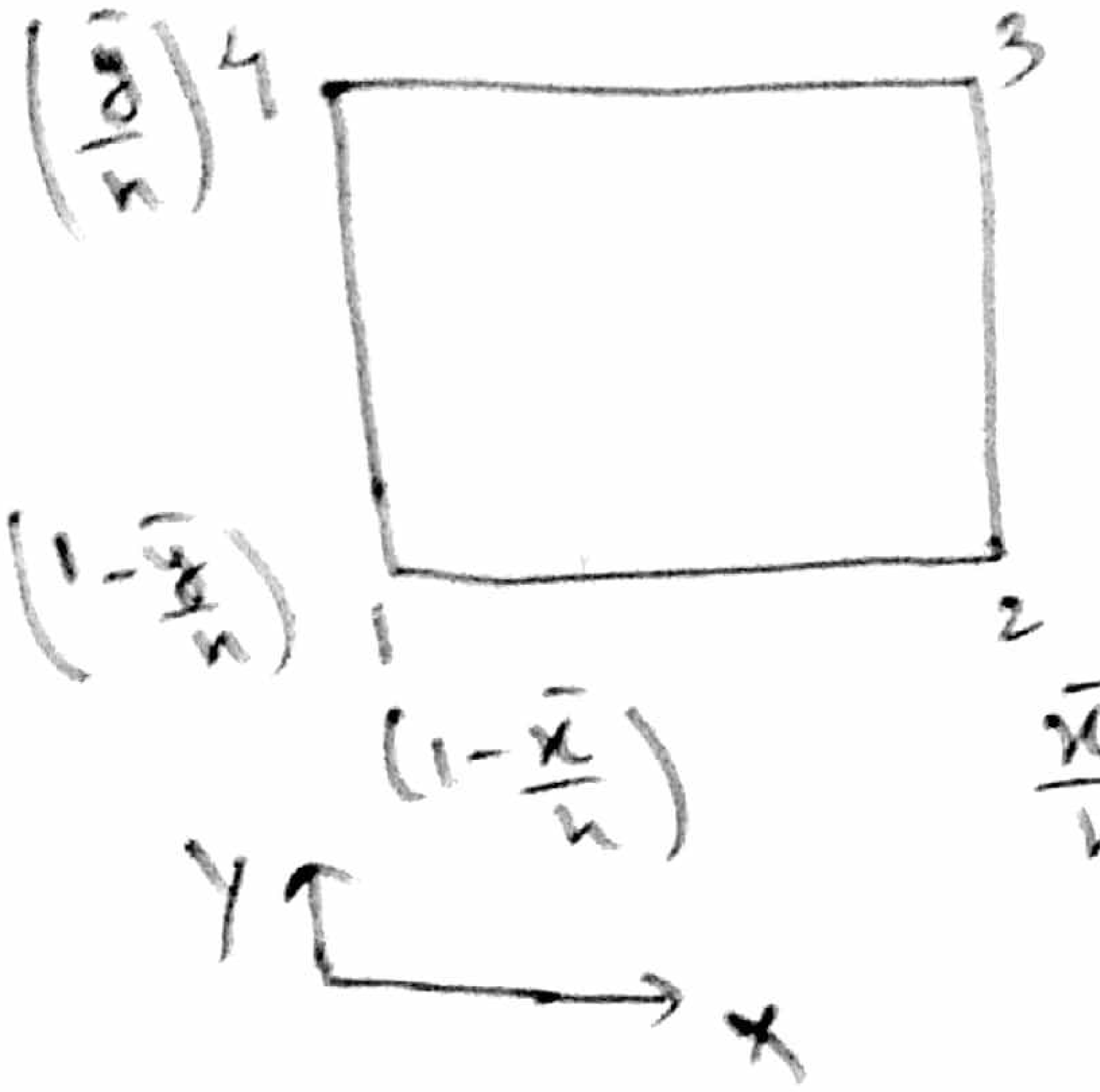
$$k_{ij} = a \int_\Omega \left(\frac{\beta_i}{2A} * \frac{\beta_j}{2A} + \frac{\gamma_i}{2A}\frac{\gamma_j}{2A} \right) dx\,dy$$

$$k_{ij} = a \left(\frac{\beta_i \beta_j + \gamma_i \gamma_j}{4A} \right)$$



1st element numbering in local coordinates was chosen arbitrarily but for other successive elements we need to follow the 1st element numbering scheme otherwise the area computation will be wrong.

Linear rectangular element



Local coordinates

$$\psi_4 = \left(1-\frac{\bar{x}}{h}\right)\frac{\bar{y}}{h}$$

$$\psi_1 = \left(1-\frac{\bar{x}}{h}\right)\left(1-\frac{\bar{y}}{h}\right)$$

$$\psi_2 = \frac{\bar{x}}{h}\left(1-\frac{\bar{y}}{h}\right)$$

$$\psi_3 = \frac{\bar{x}}{h}\,\frac{\bar{y}}{h}$$

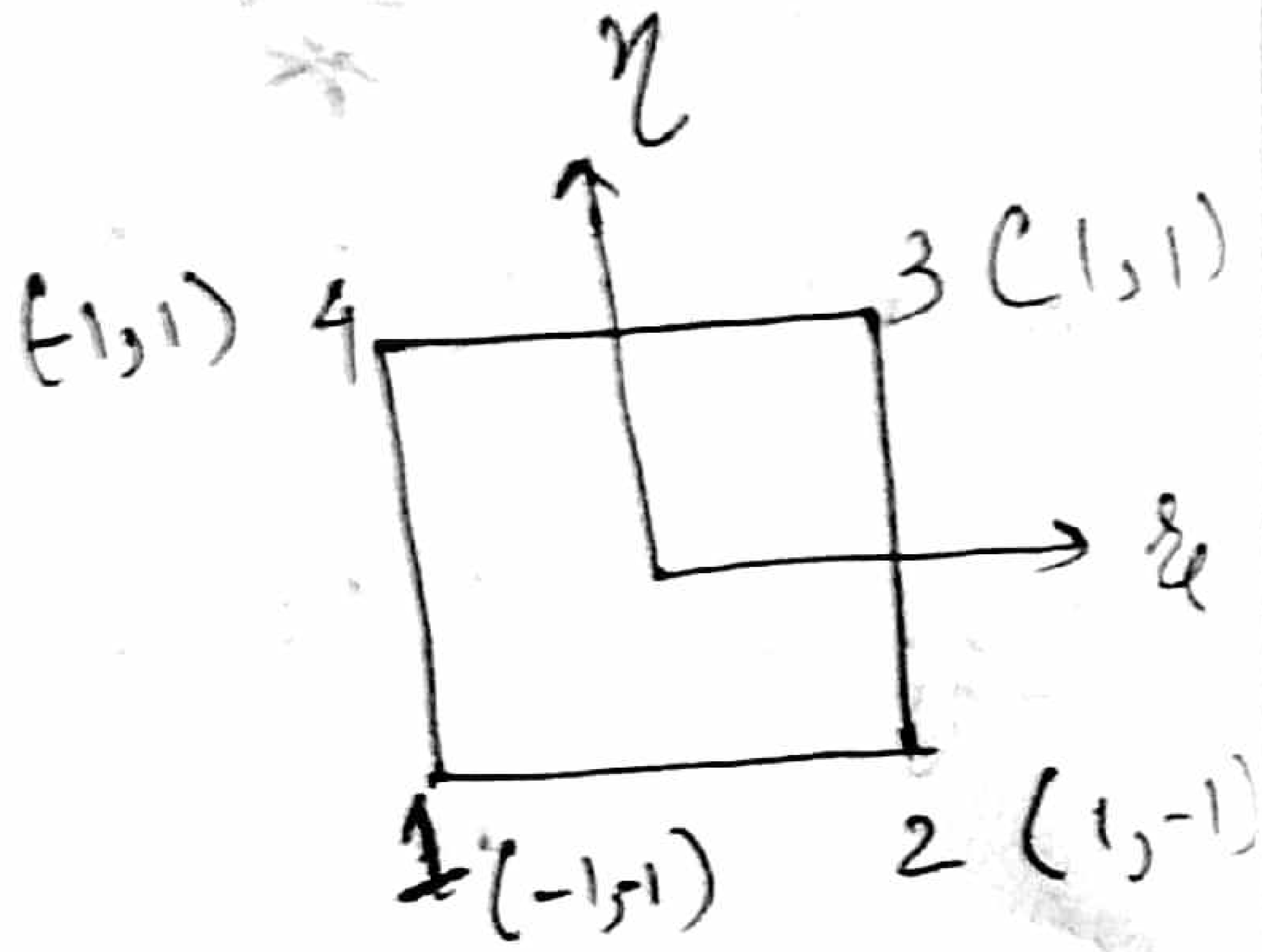
Natural coordinates

$$\psi_4 = \left(\frac{1-\xi}{2}\right)\left(\frac{1+\eta}{2}\right)$$

$$\psi_1 = \left(\frac{1-\xi}{2}\right)\left(\frac{1-\eta}{2}\right)$$

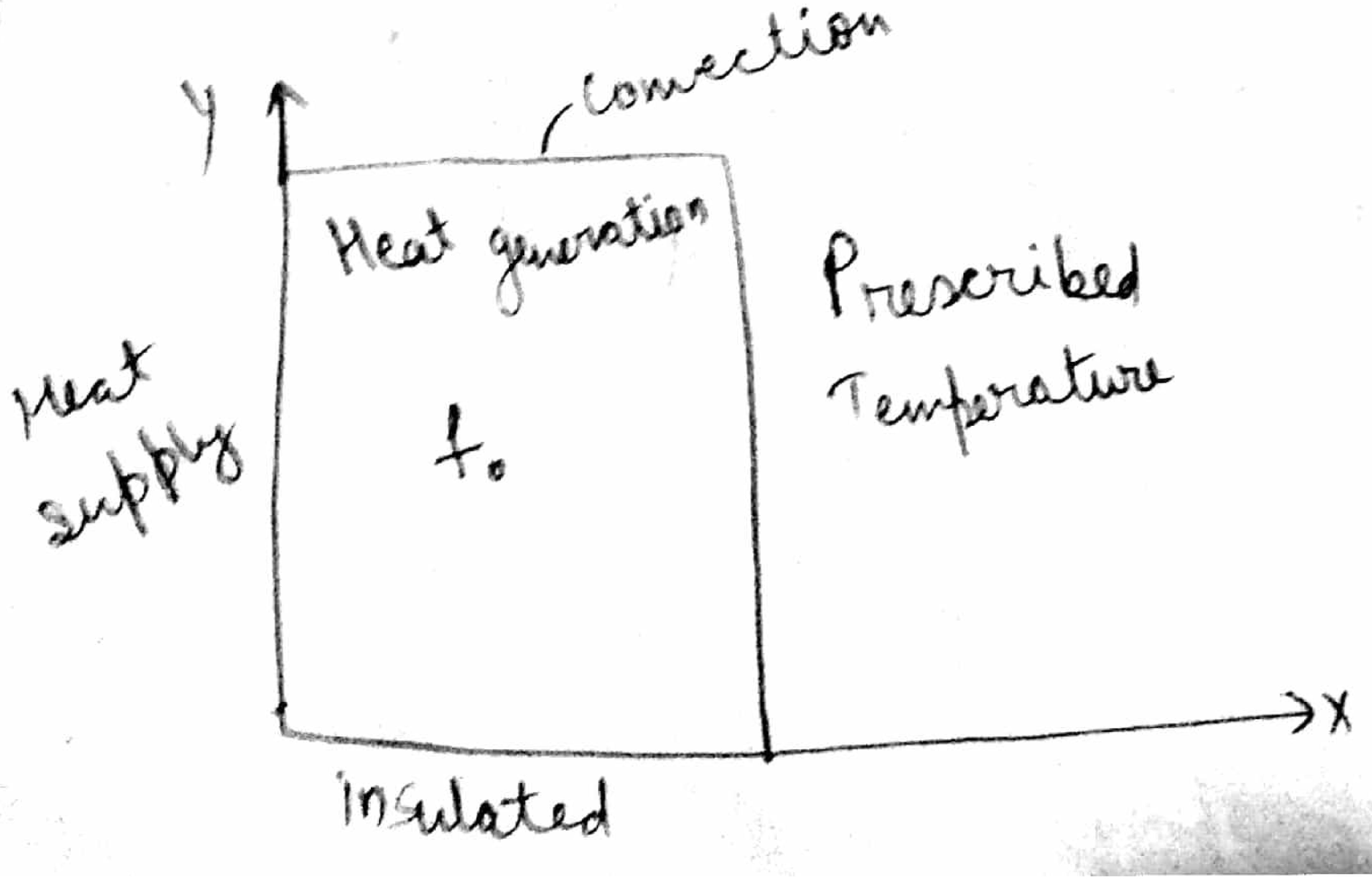
$$\psi_2 = \left(\frac{1+\xi}{2}\right)\left(\frac{1-\eta}{2}\right)$$

$$\psi_3 = \left(\frac{1+\xi}{2}\right)\left(\frac{1+\eta}{2}\right)$$



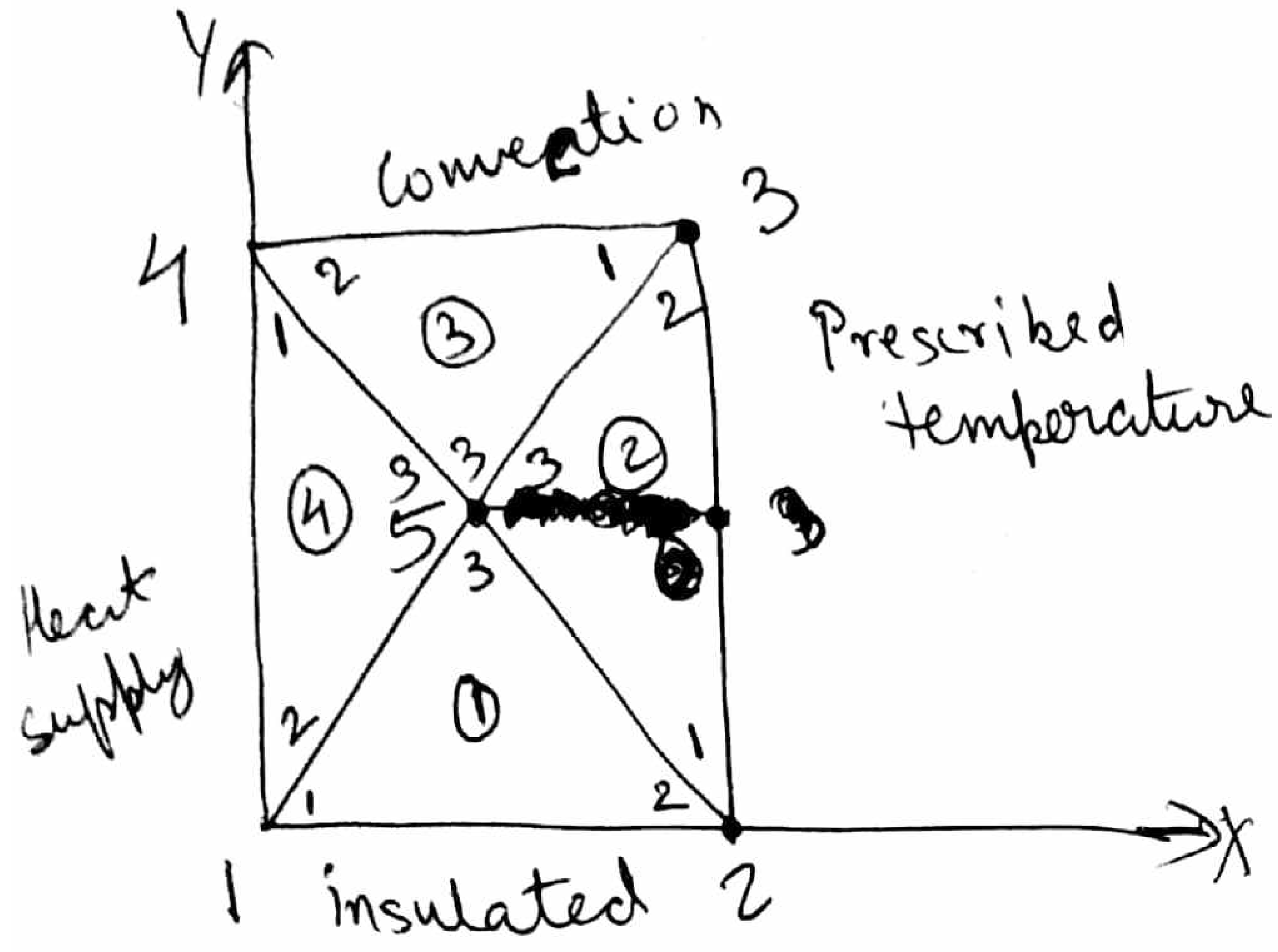
Class of interpolation functions belonging to Lagrange interpolation function.

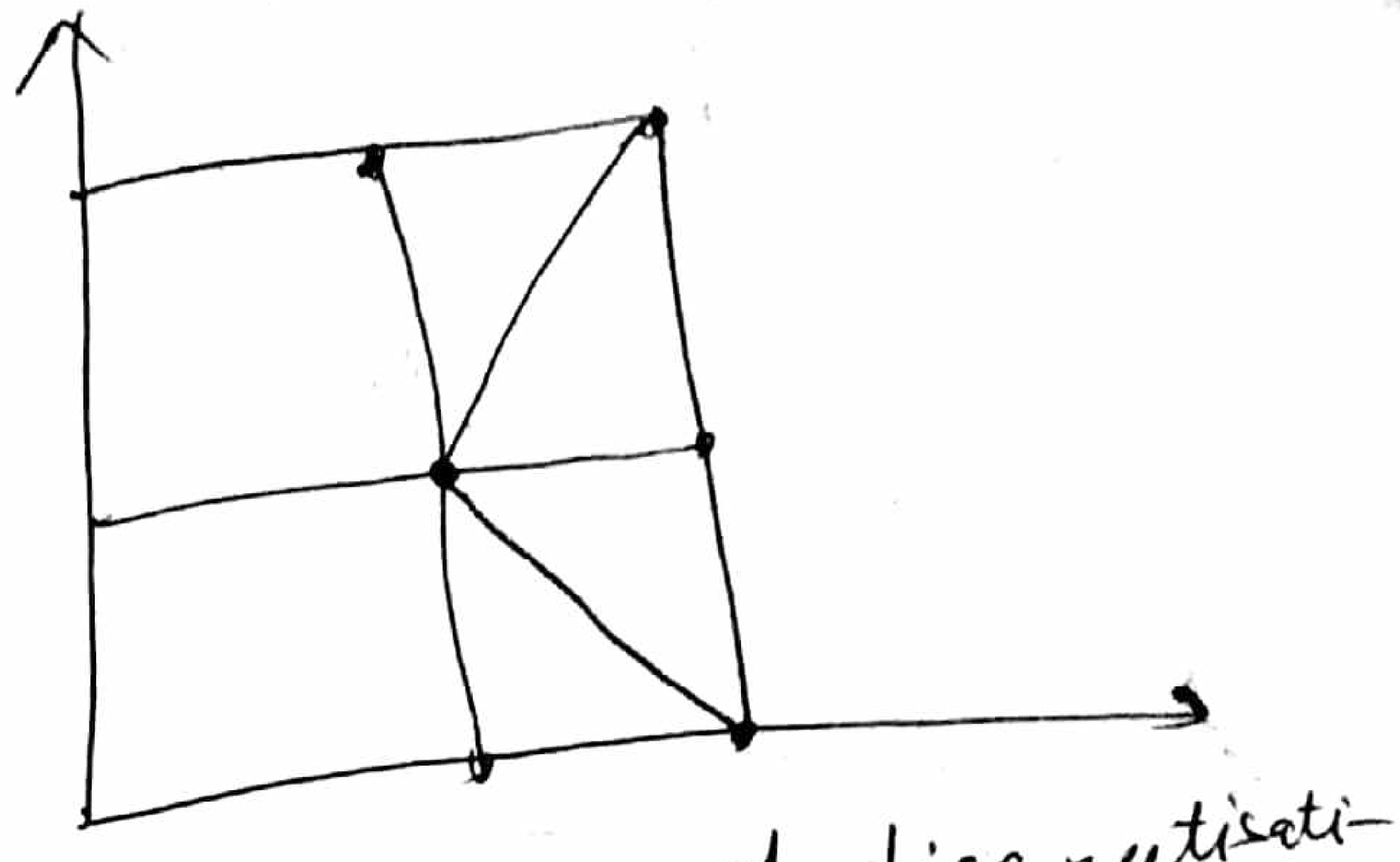
$$\int_{\Omega}\left(\psi_1^m\,\psi_2^n\,\psi_3^k\right)dx\,dy = \frac{m!\,n!\,k!}{(m+n+k+2)!}\,2A$$



$$-\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) = f(x, y)$$

$$k_x \frac{\partial T}{\partial x}\hat{n}_x + k_y \frac{\partial T}{\partial y}\hat{n}_y + \beta(T - T_\infty) = \hat{q}_n$$ ← convective boundary





This type of discretisation is also allowed

$$\int_\Omega \left(k_x \frac{\partial w}{\partial x}\frac{\partial T}{\partial x} + k_y \frac{\partial w}{\partial y}\frac{\partial T}{\partial y}\right) dx\,dy = \int_\Omega w f\,dx\,dy + \oint_s w\left(k_x \frac{\partial T}{\partial x}\hat{n}_x + k_y \frac{\partial T}{\partial y}\hat{n}_y\right) ds$$

$$= \int_\Omega w f\,dx\,dy + \oint_s w\left[\beta(T - T_\infty)\right] ds$$

$$\int_\Omega \left(k_x \frac{\partial \psi_i}{\partial x}\frac{\partial}{\partial x}\sum_{j=1}^{3} \psi_j T_j + k_y \frac{\partial \psi_i}{\partial y}\frac{\partial}{\partial y}\sum_{j=1}^{3} \psi_j T_j\right) dx\,dy$$

$$= \int \psi_i f\,dx\,dy + \oint_s \psi_i\left[\beta\left(\sum_{j=1}^{n} \psi_j T_j - T_\infty\right)\right] ds$$

$$\psi_i = \frac{1}{2A}\left(\alpha_i + \beta_i x + \gamma_i y\right)$$

$$\frac{\partial \psi_i}{\partial x} = \frac{1}{2A}\beta_i \qquad \frac{\partial \psi_i}{\partial y} = \frac{\gamma_i}{2A}$$

$$k_{ij} = \int_\Omega \left(k_x \frac{\beta_i}{2A}\frac{\beta_j}{2A} + k_y \frac{\gamma_i}{2A}\frac{\gamma_j}{2A}\right) dx\,dy$$

$$K_{ij} = \int_\Omega \frac{1}{4A^2}\left(k_x \beta_i \beta_j + k_y \gamma_i \gamma_j\right) dx\,dy$$

so, $k_{ij} = \frac{1}{4A}\left(k_x \beta_i \beta_j + k_y \gamma_i \gamma_j\right)$

$\because$ $\beta_i, \beta_j, \gamma_i, \gamma_j, k_x, k_y$ are constants Also, $\int_\Omega dx\,dy = A$

$\alpha_i = x_j y_k - x_k y_j$

$\beta_i = (y_j - y_k)$

$\gamma_i = -(x_j - x_k)$

$\alpha_1 + \alpha_2 + \alpha_3 = 2A$

Given

$x_1 = 0, \quad y_1 = 0$

$x_2 = a, \quad y_2 = 0$

$x_3 = a, \quad y_3 = a$

$x_4 = 0, \quad y_4 = a$

$x_5 = \frac{a}{2}, \quad y_5 = \frac{a}{2}$

for the first element

$\beta_1 = \left(0 - \frac{a}{2}\right) = -\frac{a}{2}$

$\beta_2 = \left(\frac{a}{2} - 0\right) = \frac{a}{2}$

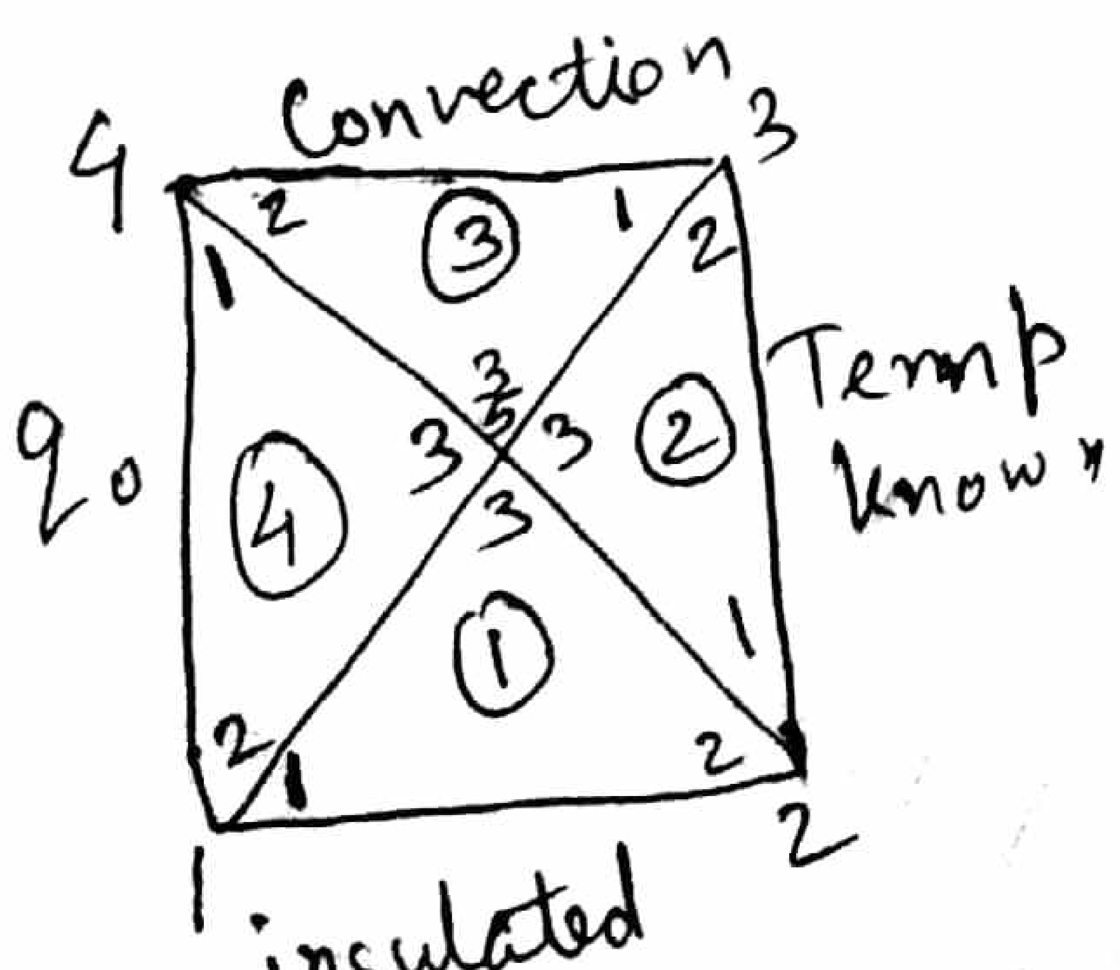
$\beta_3 = (0 - 0) = 0$

$\gamma_1 = -\left(a - \frac{a}{2}\right) = -\frac{a}{2}$

$\gamma_2 = -\left(\frac{a}{2} - 0\right) = -\frac{a}{2}$

$\gamma_3 = -(0 - a) = a$

$A = \frac{1}{2} \times a \times \frac{a}{2} = \frac{a^2}{4}$



Element	Local	Global
1	1-2-3	1-2-5
	1-2-3	2-3-5
2	1-2-3	3-4-5
3	1-2-3	4-1-5
4	1-2-3	

EBC, $T_2 = T_3 = 100$

NBC - Side 1-2 : $k_x \frac{dT}{dx} n_x^{\,0} + k_y \frac{dT}{dy} n_y \to 0$ as insulated

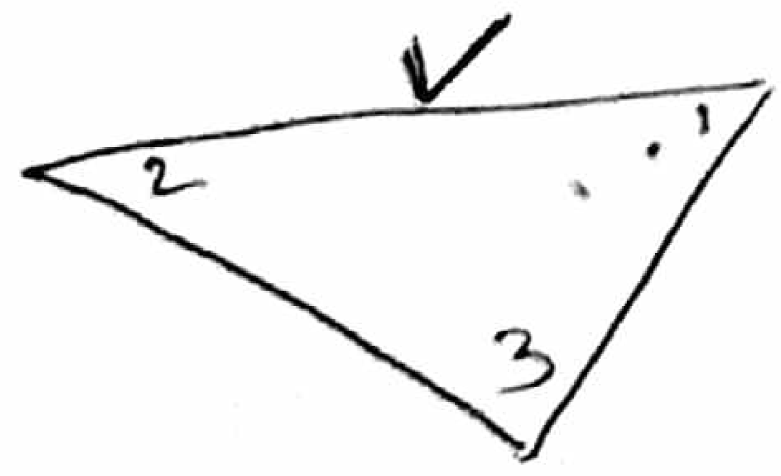
Side 3-4 $\oint w\left[q_n - \beta(T - T_\infty)\right] ds \equiv \oint \psi_i \beta T ds$ and $\int_S \psi_i \beta T_\infty ds$

q_n = External heat source on boundary 3-4 side 4-1 heat addition $= \oint \psi_i q \cdot ds$

Other integral internal heat generation

$f_i = \int_\Omega w f_0 \, dx \, dy = \int_\Omega \psi_i f_0 \, dx \, dy$ { heat generation inside the body

Local



$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

To find 'f' matrix

∵ f is on the line in this case

Hence ψ's to be put for calculating f will be of linear element Hence

This is our domain

$\alpha = \beta T_\infty$

$f_1 = \int \psi_1 \beta T_\infty \, ds = \frac{s\alpha}{2}$

$f_2 = \int \psi_2 \beta T_\infty \, ds = \frac{s\alpha}{2}$

$f_3 = \int \psi_3 \beta T_\infty \, ds = 0$

$\psi_1 = 1 - \frac{s}{h}$

$\psi_2 = \frac{s}{h}$

If  was our domain then

$\psi_1 = 1 - \frac{s}{h}$, $\psi_3 = \frac{s}{h}$

$f_1 = \int \psi_1 \beta T_\infty \, ds = \frac{s\alpha}{2}$

$f_2 = \int \psi_2 \beta T_\infty \, ds = 0$

$f_3 = \int \psi_3 \beta T_\infty \, ds = \frac{s\alpha}{2}$

$\oint \psi_i \beta \sum_{j=1}^{3} T_j \psi_j \, ds$

$$\begin{bmatrix} \overset{C_{11}}{\int \psi_1 \psi_1} & \overset{C_{12}}{\int \psi_1 \psi_2} & \cancel{\int \psi_1 \psi_3} \to 0 \\ \overset{C_{21}}{\int \psi_1 \psi_2} & \overset{C_{22}}{\int \psi_2 \psi_2} & \cancel{\int \psi_2 \psi_3} \to 0 \\ \cancel{\int \psi_1 \psi_3} & \cancel{\int \psi_2 \psi_3} & \cancel{\int \psi_3 \psi_3} \to 0 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

$$\begin{bmatrix} K_{11}+C_{11} & K_{12}+C_{12} & K_{13} \\ K_{21}+C_{21} & K_{22}+C_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \begin{matrix} \to \frac{\beta T_\infty h}{2} \\ \to \frac{\beta T_\infty h}{2} \\ \to 0 \end{matrix}$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \frac{\beta h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

For q_0 boundary

$$f_1 \to \frac{q_0 h}{2}$$

$$f_2 \to \frac{q_0 h}{2}$$

$$f_3 \to 0$$

For internal heat generation boundary condition.

$$\int_\Omega \psi_i f_0 \, dx\, dy = \frac{f_0 A}{3}$$

As it is integration over domain
ψ_i here is triangular element interpolation function

$$\int_\Omega \psi^m \psi^n \psi^p \, d\Omega = \frac{m!\, n!\, p!}{(m+n+p+2)!} \times 2A$$

Element $K_{ij} = \dfrac{k_x \beta_i \beta_j + k_y \beta_i \beta_j}{4A}$

$$[K] = \frac{k}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

↑ local

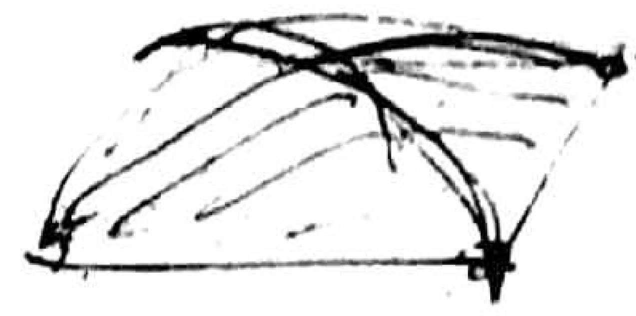
$[K]$ = (global)

	1	2	3	4	5
1	$k^1_{11} + k^4_{22}$	k^1_{12}	0	k^4_{21}	$k^1_{13} + k^4_{23}$
2	k^1_{21}	$k^1_{22} + k^2_{11}$	k^2_{12}	0	$k^1_{23} + k^2_{13}$
3	0	k^2_{21}	$k^2_{22} + k^3_{11}$	k^3_{12}	$k^2_{23} + k^3_{13}$
4	k^4_{12}	0	k^3_{21}	$k^4_{11} + k^3_{22}$	$k^4_{13} + k^3_{23}$
5	$k^1_{31} + k^4_{32}$	$k^1_{32} + k^2_{31}$	$k^2_{32} + k^3_{31}$	$k^4_{31} + k^3_{32}$	$k^1_{33} + k^2_{33} + k^4_{33} + k^3_{33}$

$$k^1_{33} + k^2_{33} + k^4_{33} + k^3_{33} = (2+2+2+2)\frac{k}{2} = 4k$$

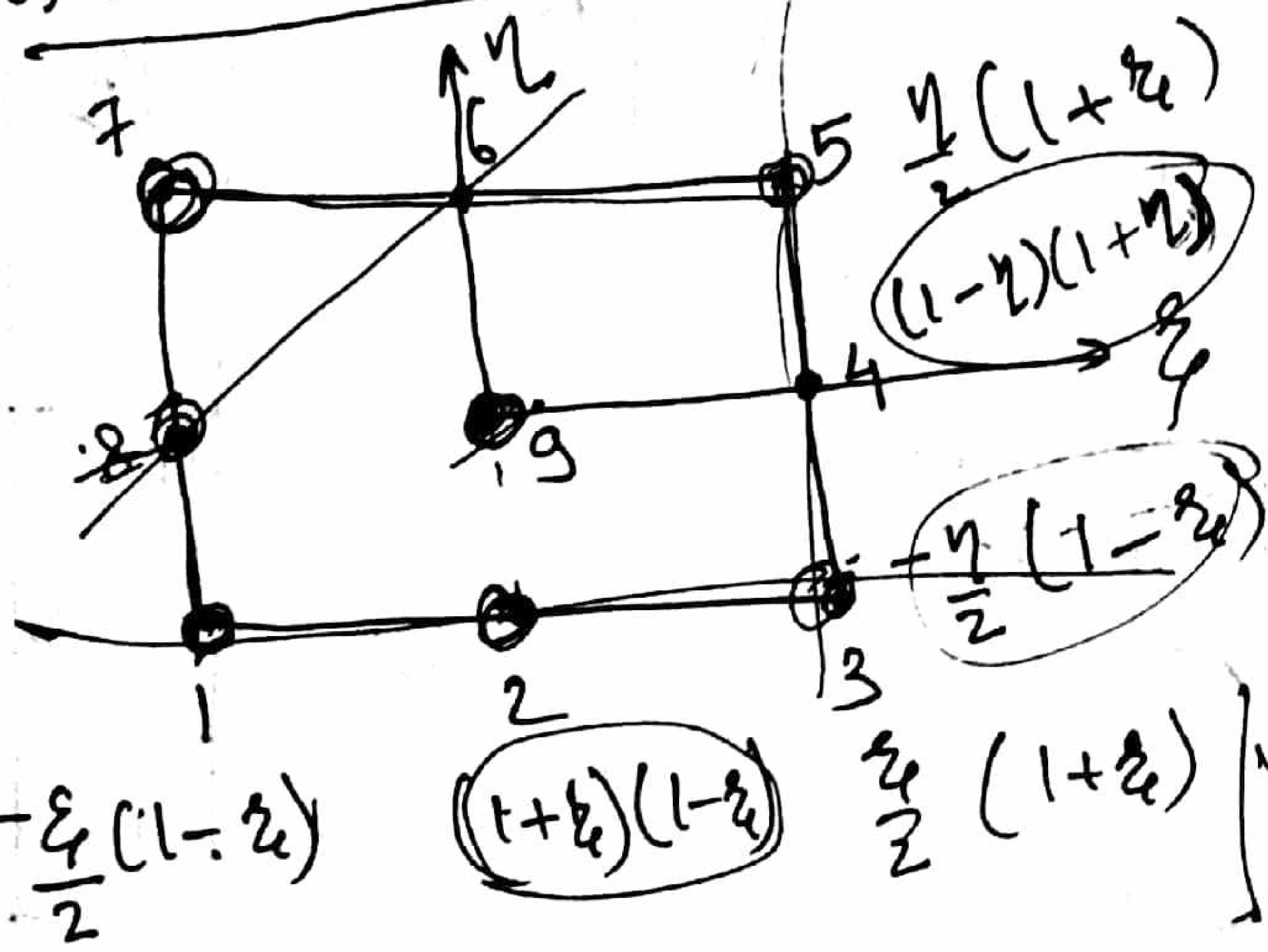
$$[k] \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{Bmatrix} f_1^1 + f_2^4 \\ f_2^1 + f_1^2 \\ f_2^3 + f_1^3 \\ f_2^3 + f_1^4 \\ f_3^1 + f_3^2 + f_3^3 + f_3^4 \end{Bmatrix}$$

↑ Global



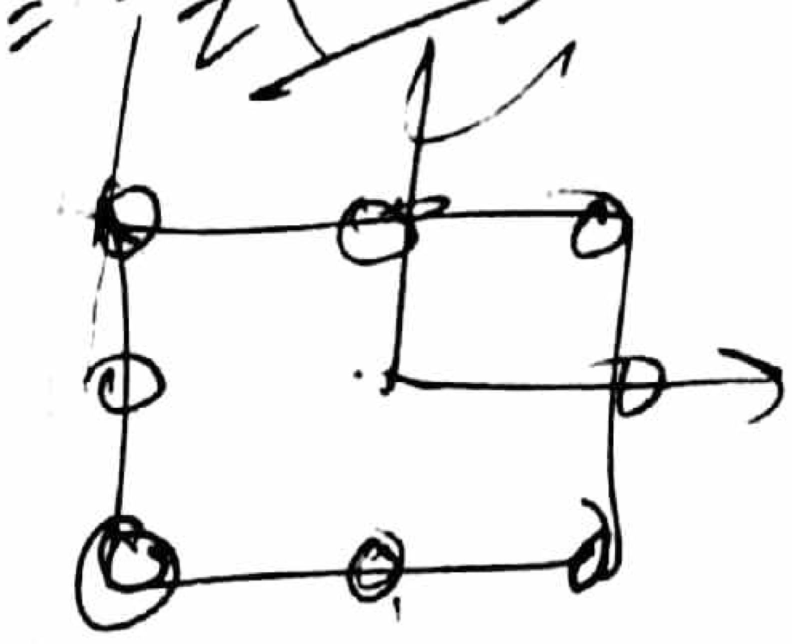
Interpolation Function

Quadratic interpolation (rectangular)



$\psi_8 = (1-\xi^2)(1-\eta)$

$\psi_8 = -\frac{\xi}{2}(1-\xi)(1-\eta)$



$$\psi_9 = (1-\xi)(1+\xi)(1-\eta)(1+\eta)$$

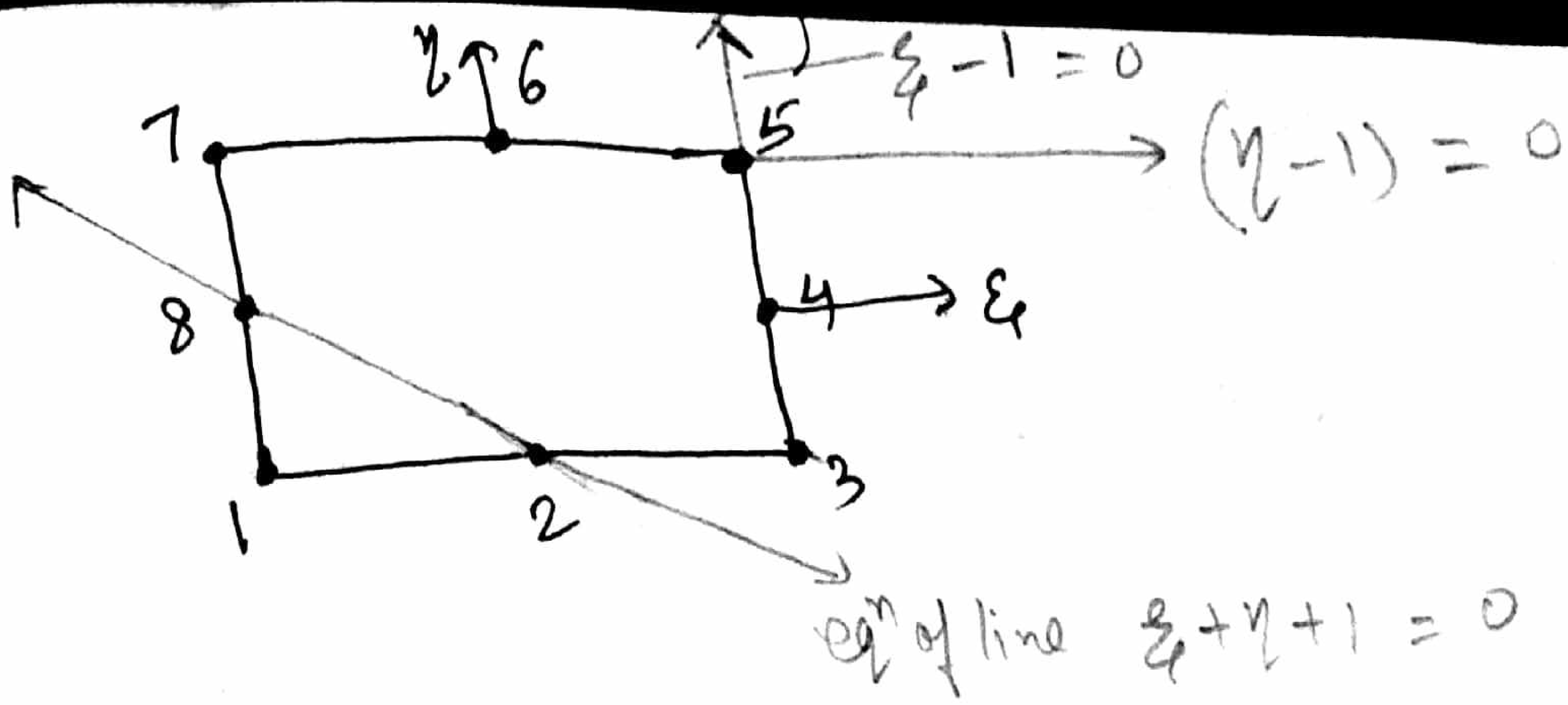
$$\psi_9 = (1-\xi^2)(1-\eta^2)$$

$$\psi_3 = \frac{\xi}{2}(1+\xi)\left(-\frac{\eta}{2}\right)(1+\eta) = -\frac{\eta\xi}{4}(1+\xi)(1+\eta)$$

This particular with 8 node number element is very common in Ansys & abacus and is called serendiety element.

How do you evaluate the problem with 8 node points.

Polynomial method → By using the properties of interpolation function we derieve the interpolation function.



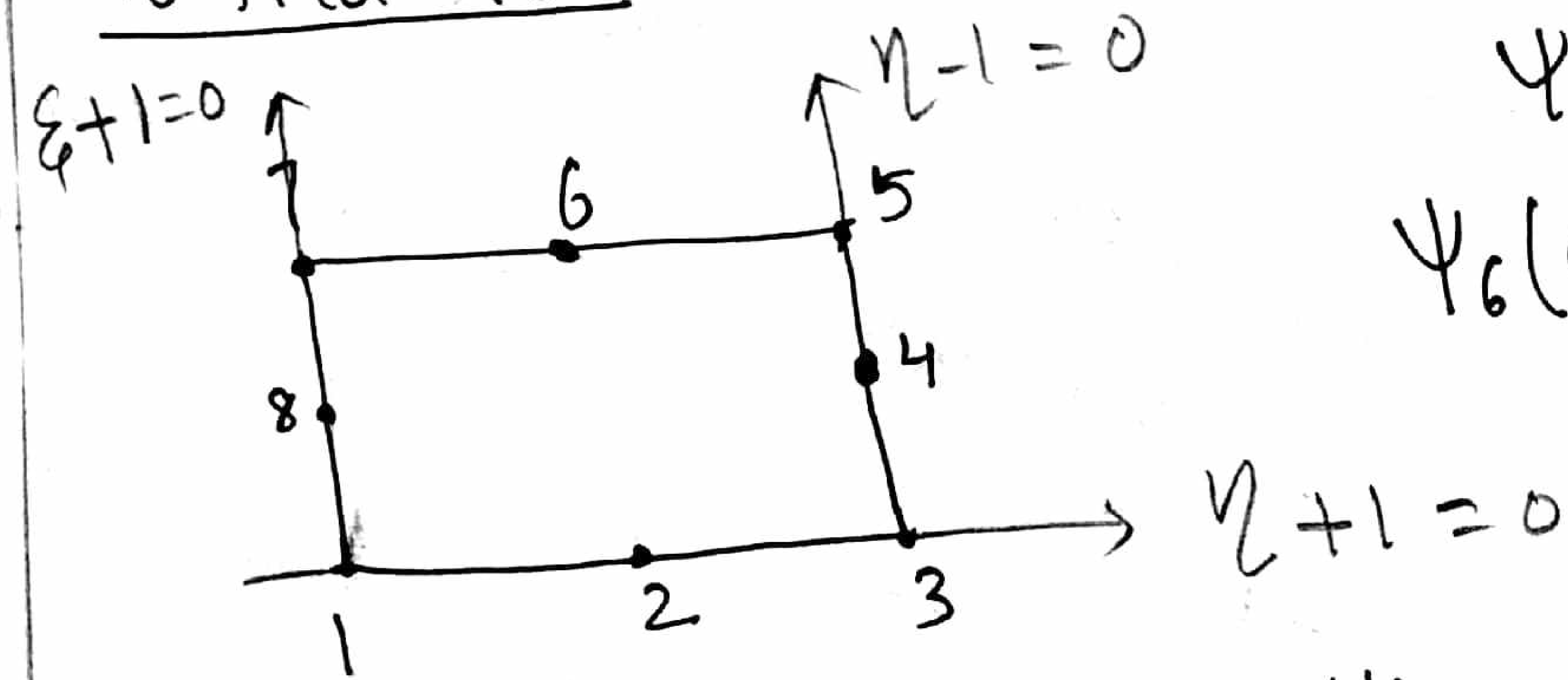
$$\psi_1 = c_1(\xi + \eta + 1)(\xi - 1)(\eta - 1)$$

at $(\xi, \eta) = (-1, -1)$

$$\psi_1(-1,-1) = 1 = c_1(-1-1+1)(-1-1)(-1-1)$$

$$\Rightarrow \boxed{c_1 = -\frac{1}{4}}$$

To find ψ_6



$$\psi_6 = c_6(\xi + 1)(\eta + 1)(\eta - 1)$$

$$\psi_6(0,1) = c_6(1)(2)(-1)$$

$$\boxed{c_6 = -\frac{1}{2}}$$

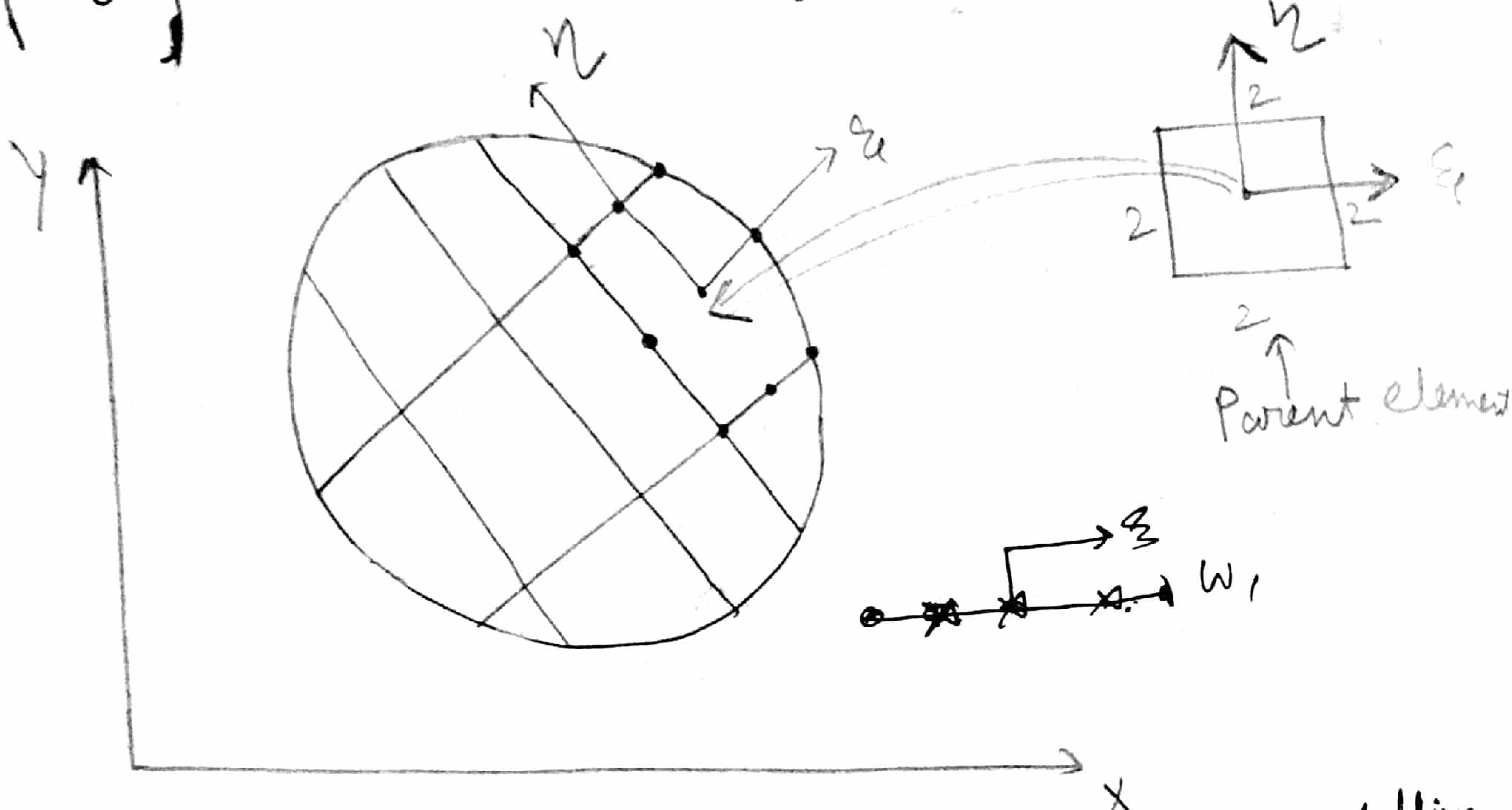
$$\psi_6 = -\frac{1}{2}(\eta^2 - 1)(1 + \xi)$$

$$\frac{\partial \psi_i}{\partial \xi} = \frac{\partial \psi_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \psi_i}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial \psi_i}{\partial \eta} = \frac{\partial \psi_i}{\partial \eta}\frac{\partial x}{\partial \eta} + \frac{\partial \psi_i}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\begin{Bmatrix} \frac{\partial \psi_i}{\partial \xi} \\ \frac{\partial \psi_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial \psi_i}{\partial x} \\ \frac{\partial \psi_i}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial \psi_i}{\partial x} \\ \frac{\partial \psi_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial \psi_i}{\partial \xi} \\ \frac{\partial \psi_i}{\partial \eta} \end{Bmatrix}$$



← Geometric modelling

$$x = x_1\psi_1 + x_2\psi_2 + \cdots + x_9\psi_9$$

$$y = y_1\psi_1 + y_2\psi_2 + \cdots + y_9\psi_9$$

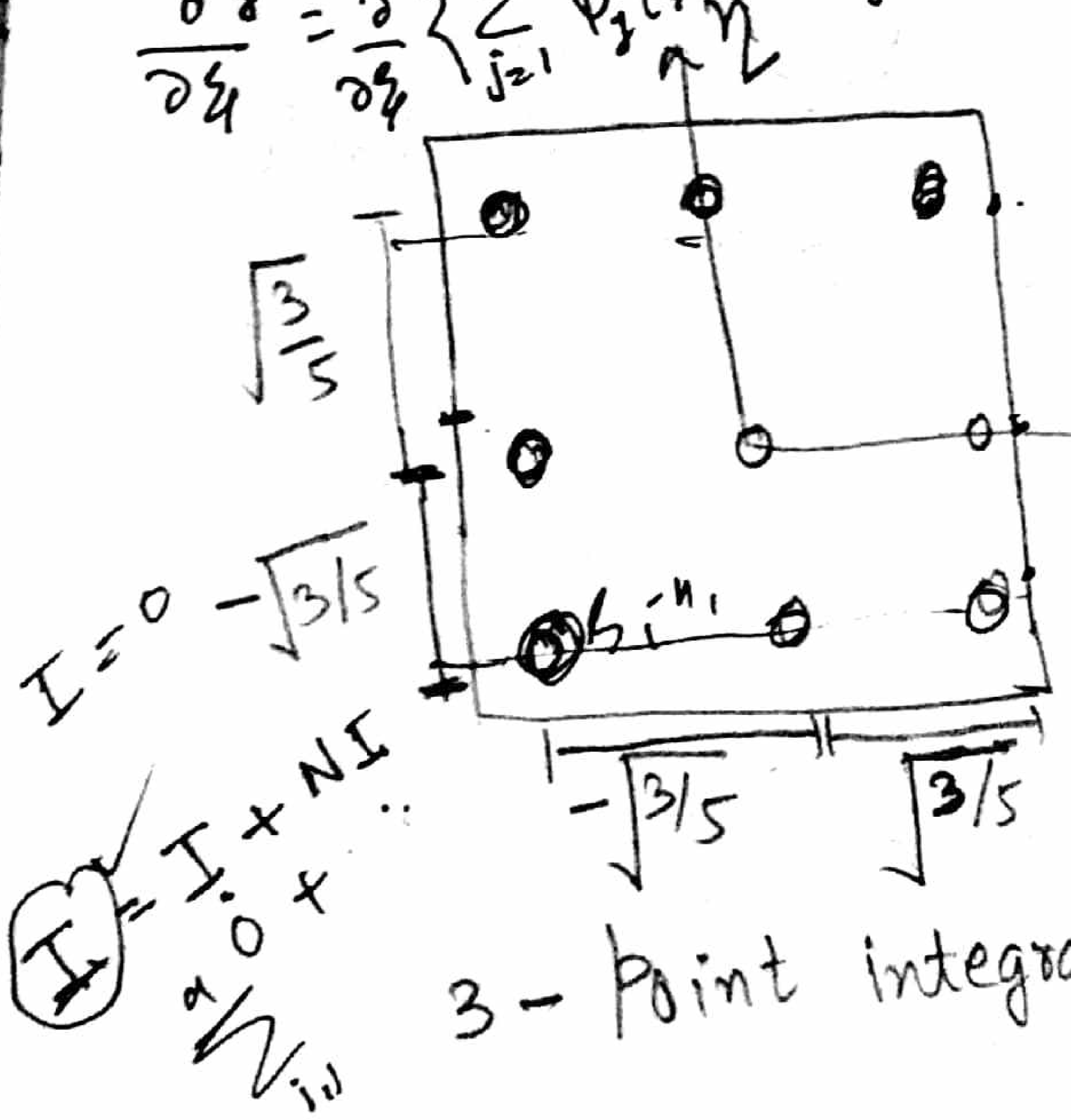
$$\frac{\partial y}{\partial \xi} = \frac{\partial}{\partial \xi}\left\{\sum_{j=1}^{n} \psi_j(\xi,\eta)\, y_j\right\}$$

$$\int_{-1}^{+1} f(\xi,\eta)\, d\xi\, d\eta = f_{center} \times W_i \times W_j$$

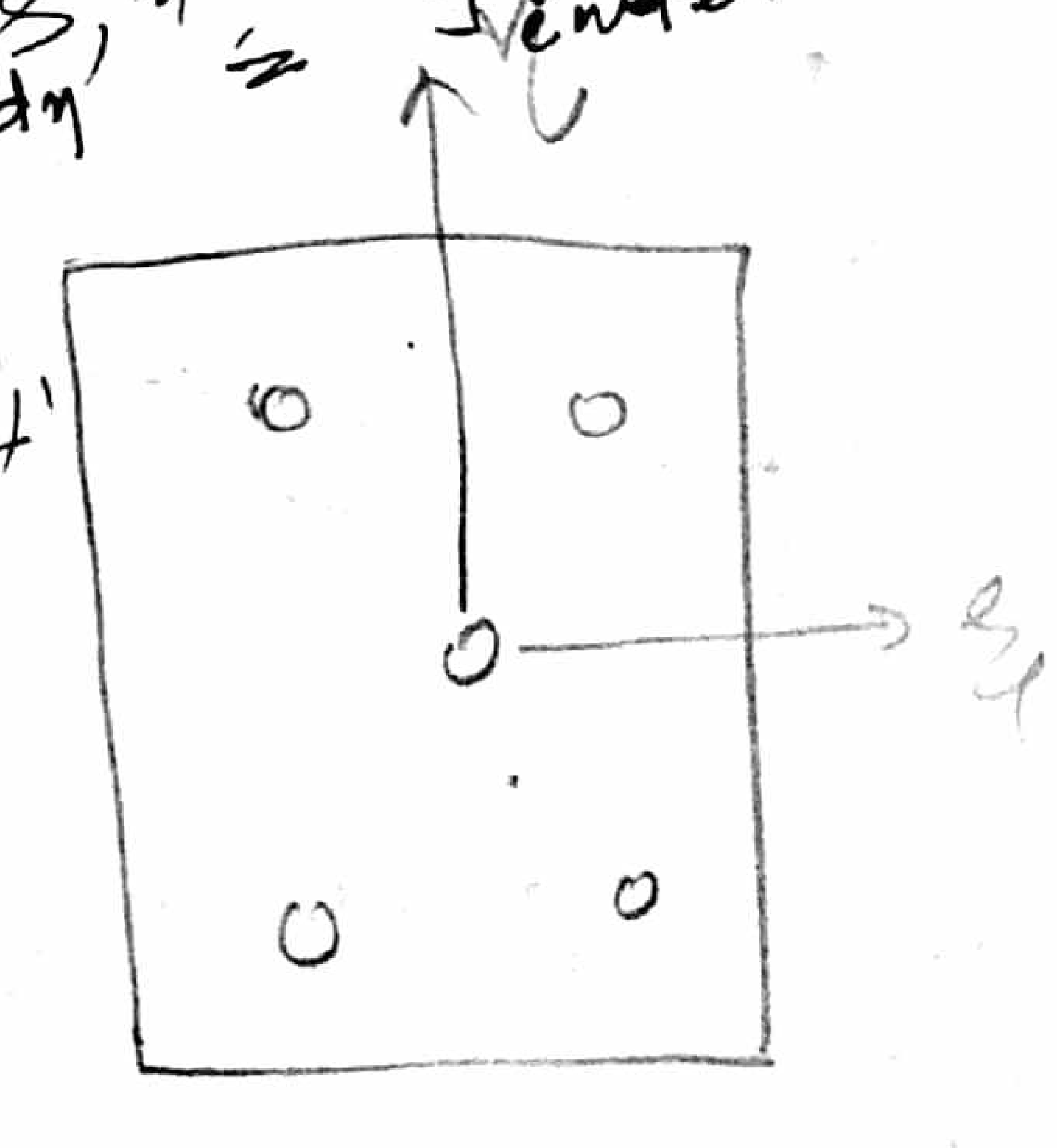
$$\sum_{\xi j=1}^{n} \ldots$$

(ξ_i, η_j)

$-\sqrt{3/5} \;\; \sqrt{3/5}$



3 - Point integration.



2 - point integration

Functions are to be evaluated at ξ_i & η_j so integrand will be $f(\xi_i, \eta_j) * w_i * w_j$

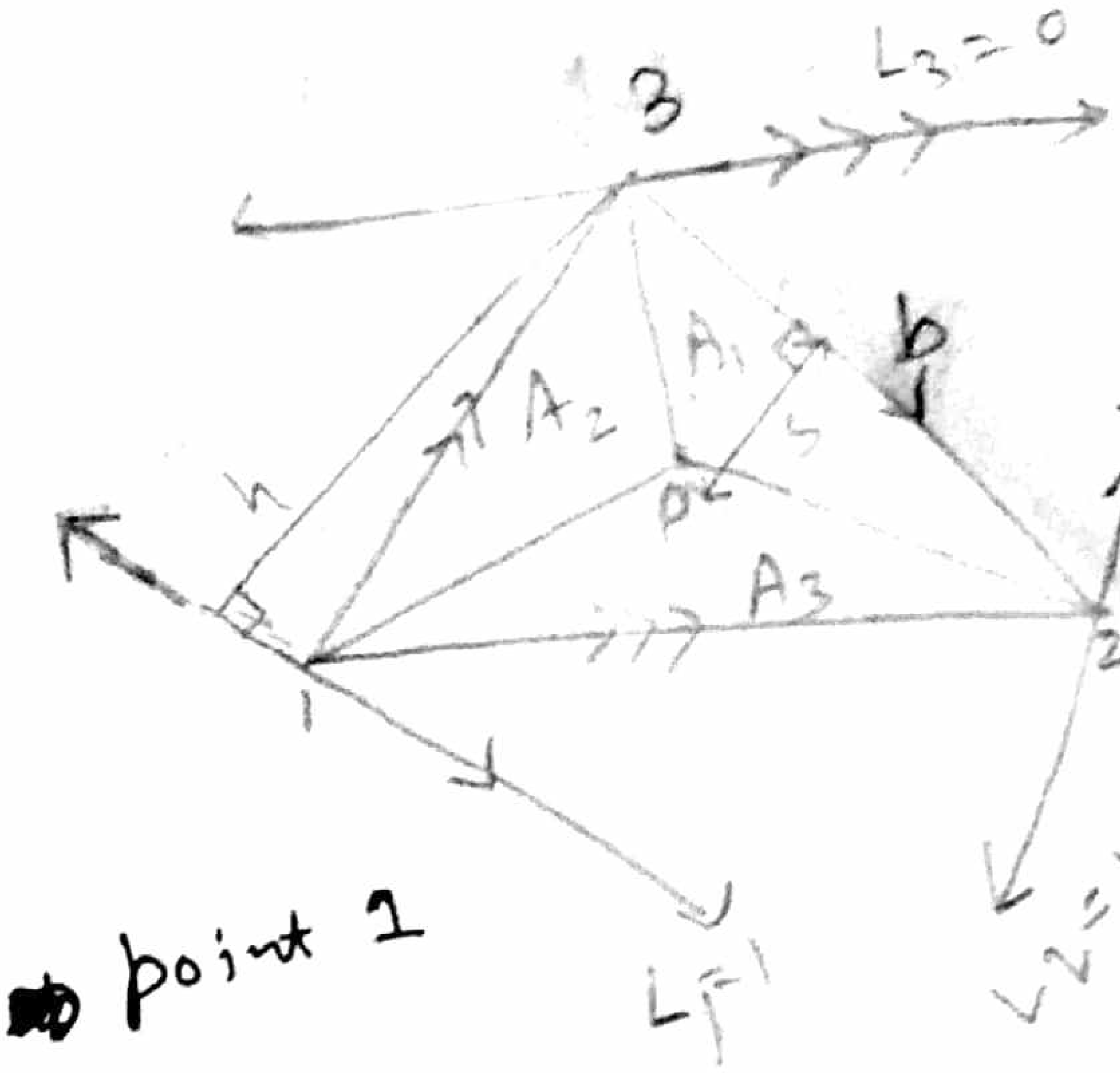
Area coordinates

$A_1 + A_2 + A_3 = A$

$A_1 = \frac{1}{2} b s$

$A = \frac{1}{2} b h$

$\frac{A_1}{A} = \frac{s}{h} = L_1$



h is ⊥ distance between 2-3 and point 1

b = length of side 2 - 3

s = ⊥ distance b/w side 2 - 3 and P

Quadratic element

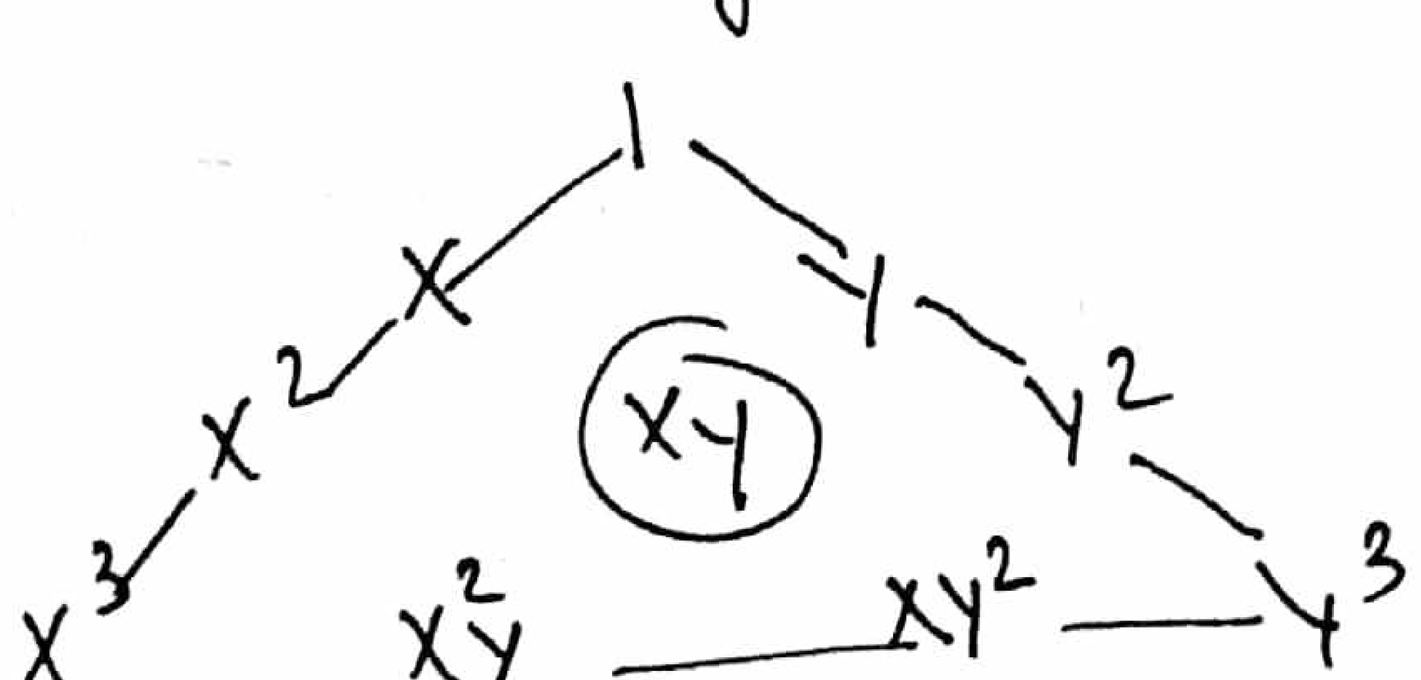
$\Psi_1 = C_1 \left(L_1 - \frac{1}{2}\right)\left(L_1 - 0\right)$

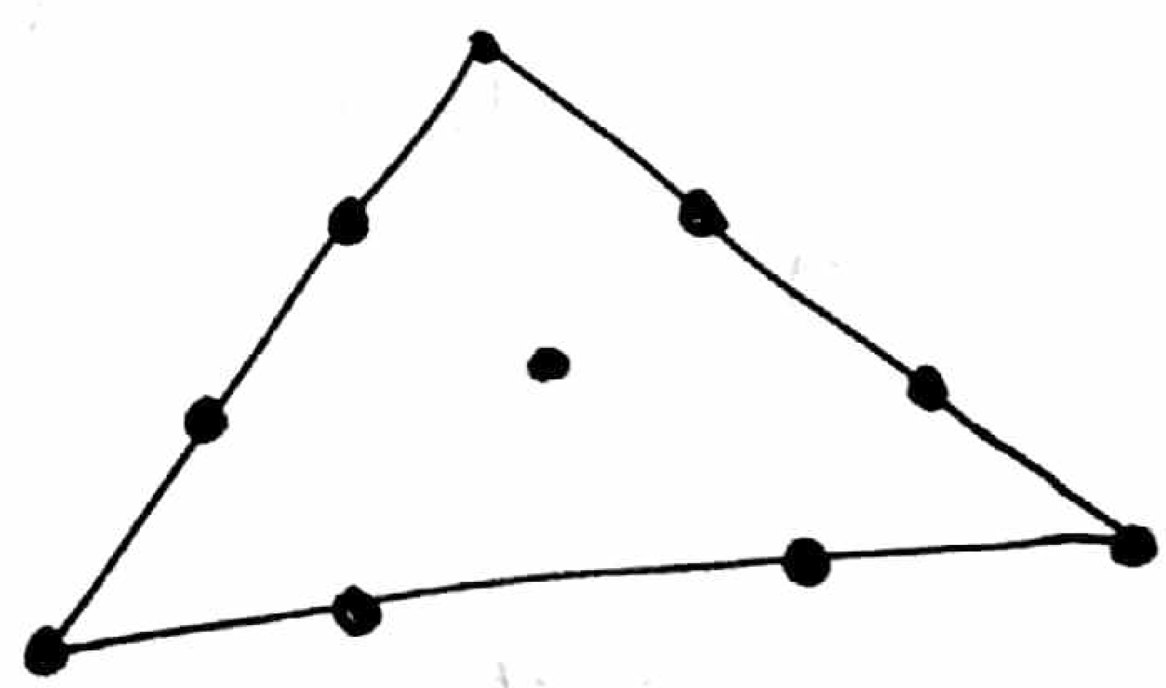
$1 = C_1 \left(1 - \frac{1}{2}\right)(1 - 0) \Rightarrow \boxed{C_1 = 2}$

$\Rightarrow \Psi_1 = L_1 (2L_1 - 1)$

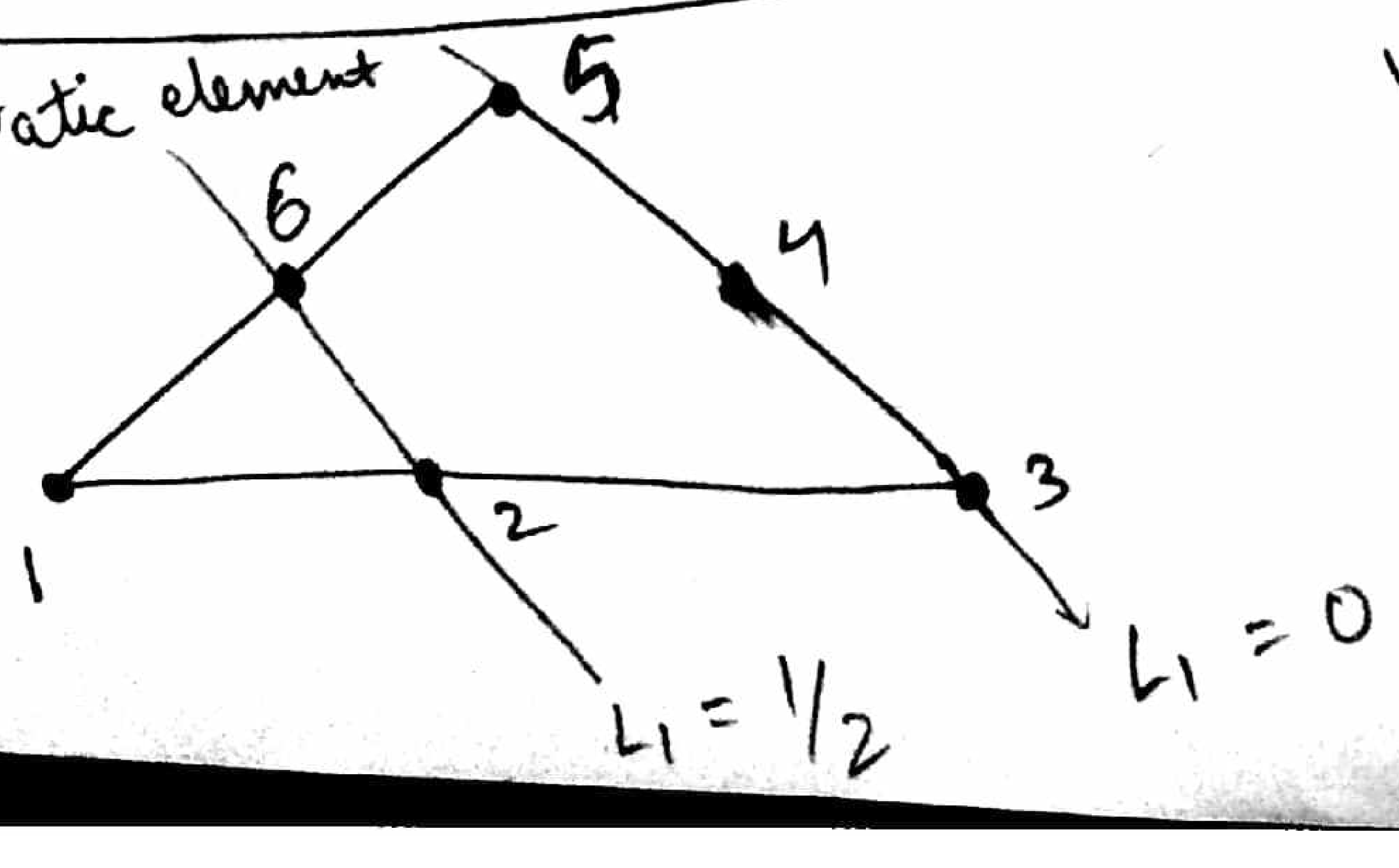
$L_1 + L_2 + L_3 = 1$, L_1, L_2, L_3 are called area coordinates.

Notes:- Cubic triangular element





Quadratic element



$\Psi_1 = C_1 L_1 \left(L_1 - \frac{1}{2}\right)$

$1 = C_1 (1) \left(\frac{1}{2}\right)$

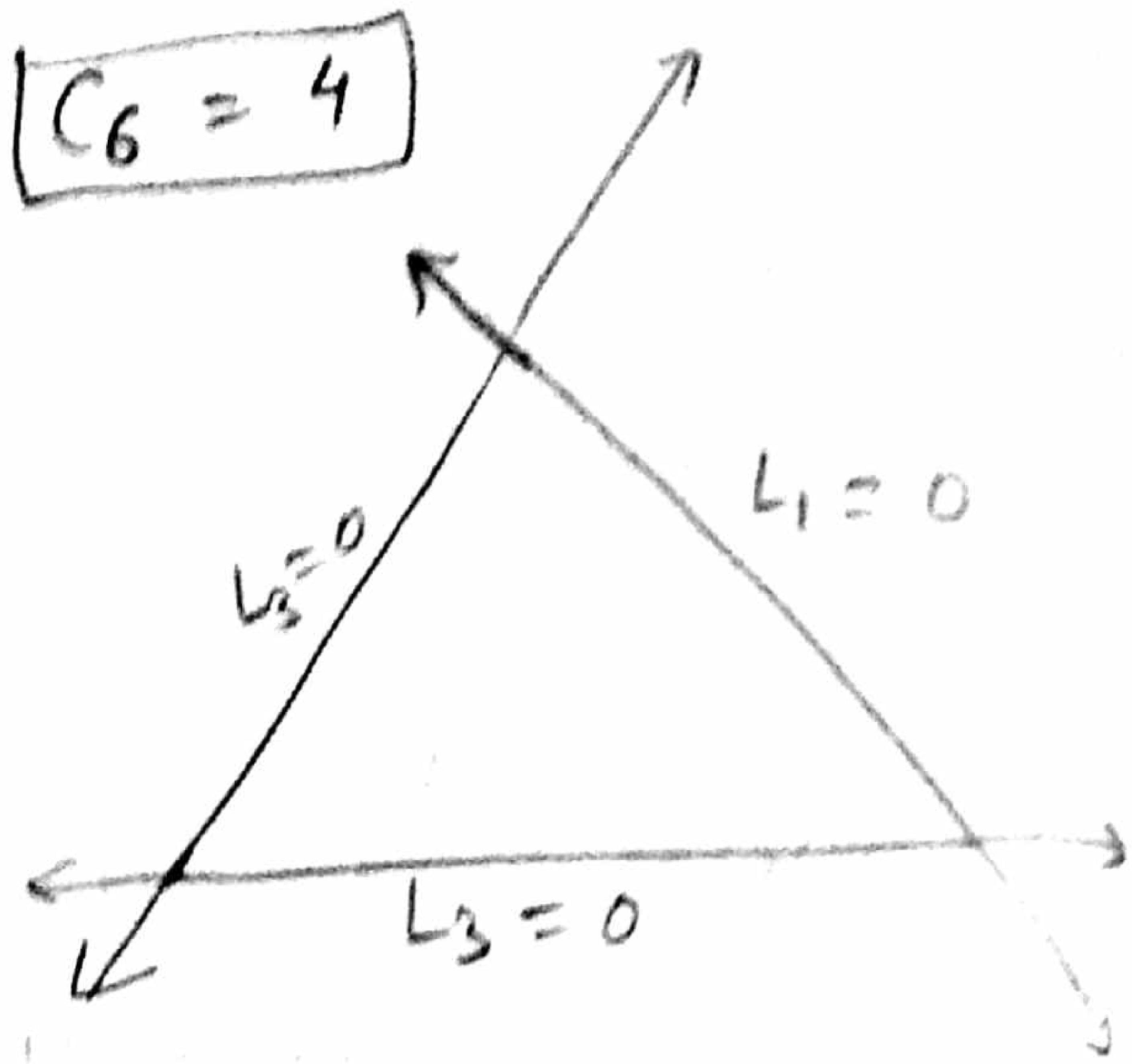
$\Rightarrow \boxed{C_1 = 2}$

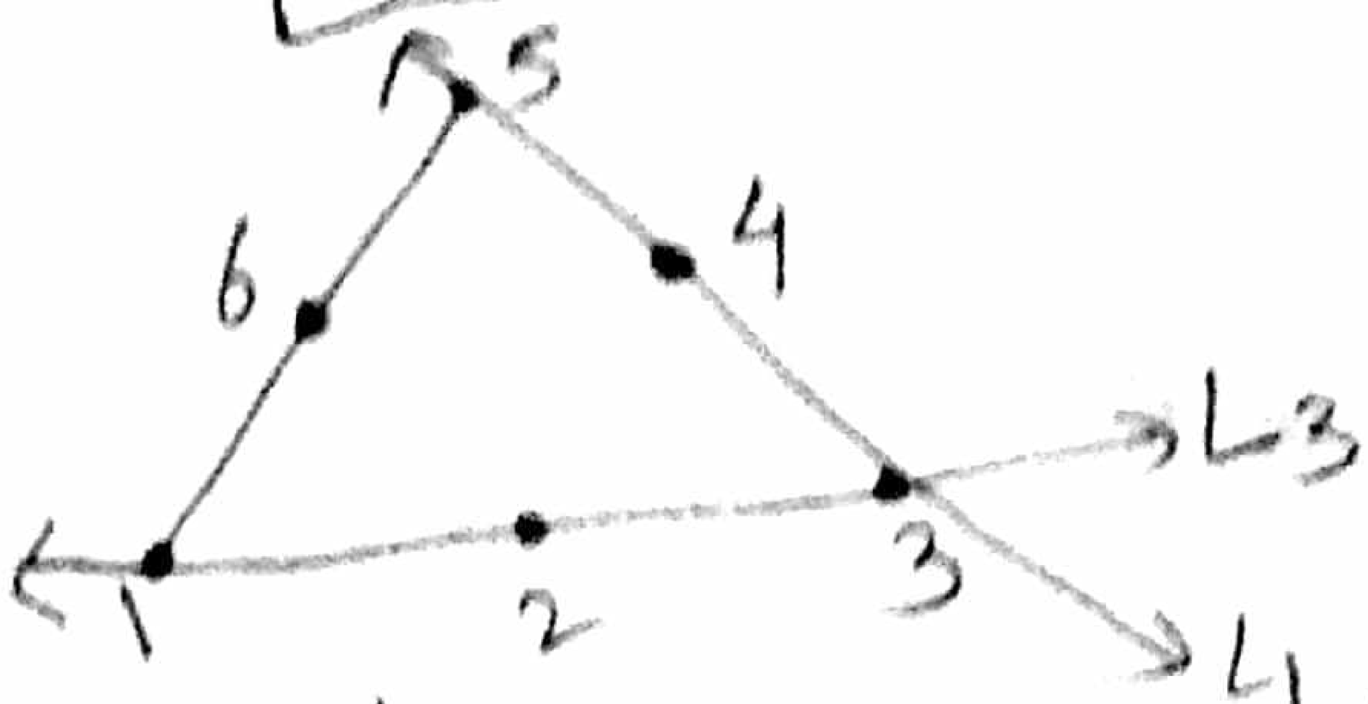
$\boxed{\Psi_1 = L_1 (2L_1 - 1)}$

$$\psi_6 = c_6 * L_1 * L_3$$

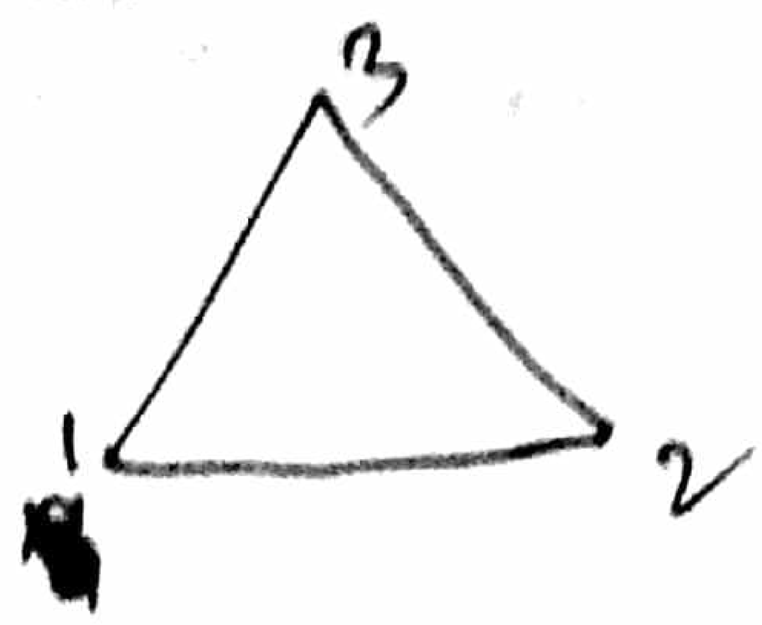
$$1 = c_6 * \frac{1}{2} * \frac{1}{2} \Rightarrow \boxed{c_6 = 4}$$

$$\boxed{\psi_6 = 4 L_1 L_3}$$





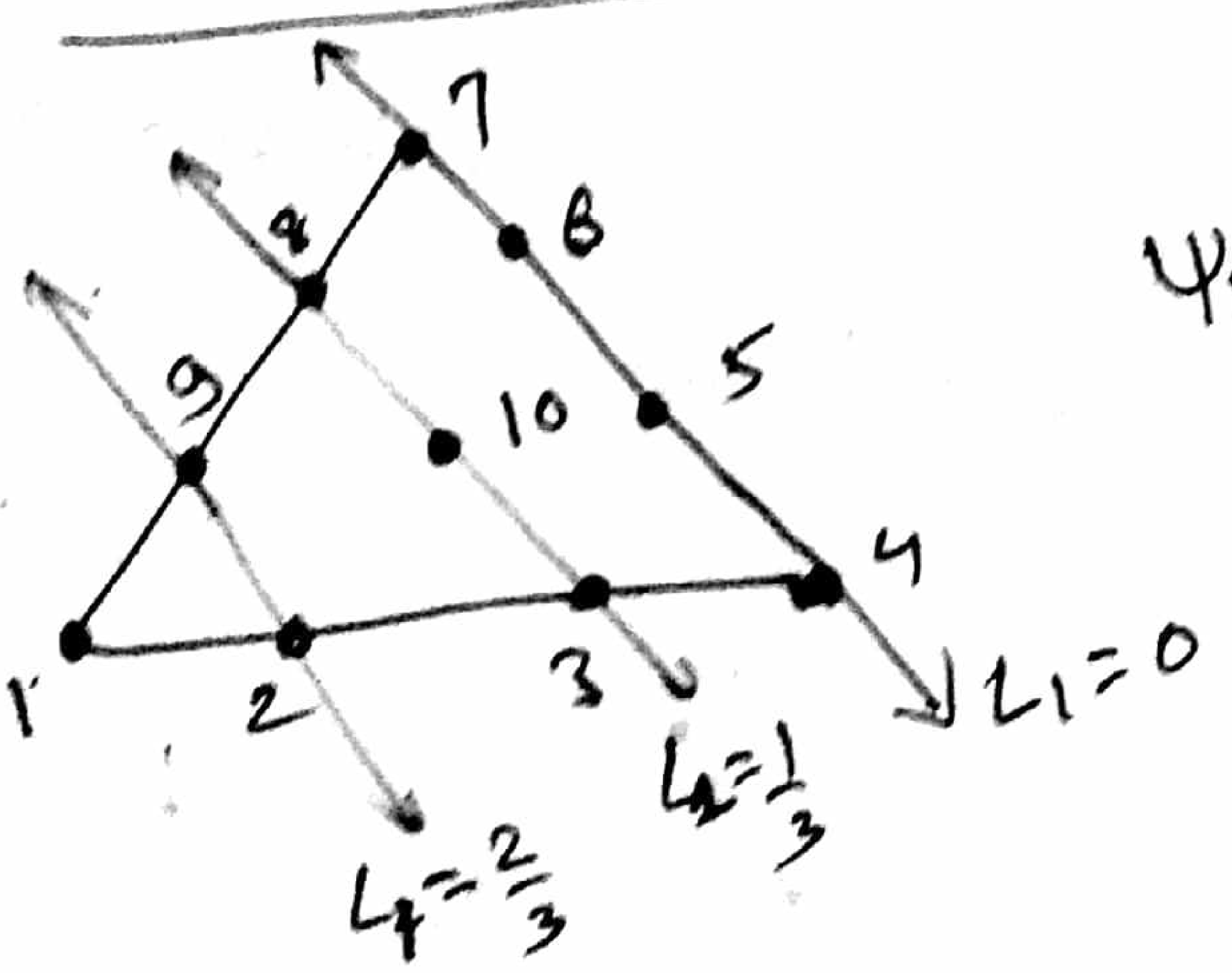
For linear triangular element



$$\psi_1 = c_1 L_1$$

$$c_1 = 1 \Rightarrow \boxed{\psi_1 = L_1}$$

For cubic triangular element



$$\psi_1 = c_1 \left(L_1 - \frac{2}{3}\right)\left(L_1 - \frac{1}{3}\right) L_1$$

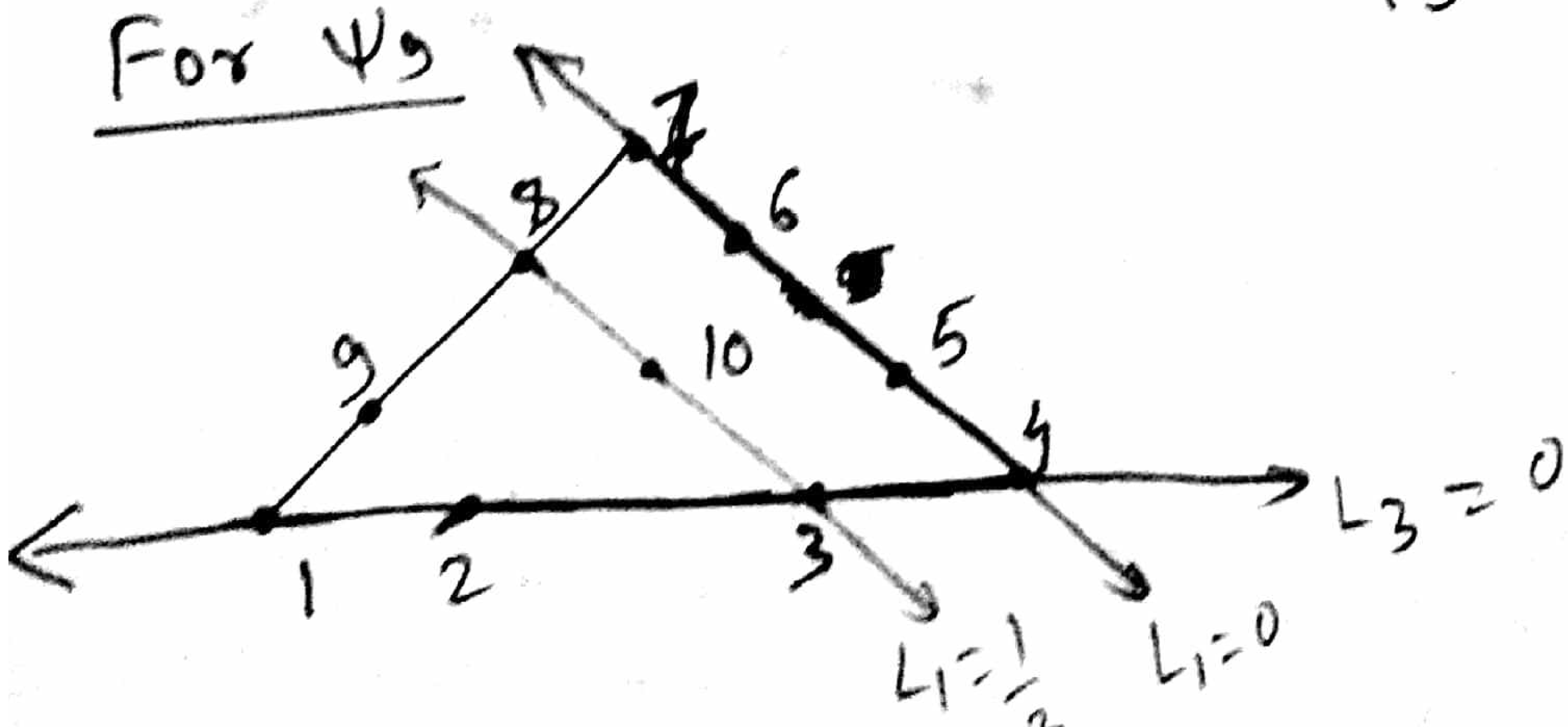
$$\psi_1(1) = c_1 (1/3)(2/3)(1)$$

$$c_1 = 9/2$$

$$\psi_1 = \frac{(3L_1 - 2)(3L_1 - 1) L_1}{2}$$

$$\psi_9 = c_1 L_1 \left(L_1 - \frac{1}{3}\right) L_3$$

For ψ_9



For triangular element

$$\psi_1 = f(L_1, L_2, L_3)$$

$$L_1 + L_2 + L_3 = 1 \Rightarrow L_3 = 1 - L_1 - L_2$$

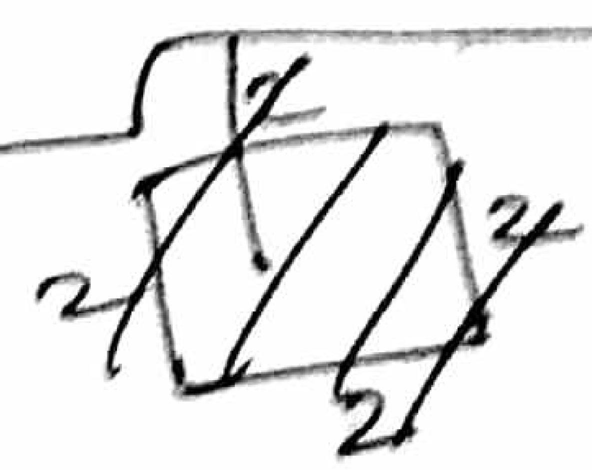
$$\psi_1 = f(L_1, L_2, 1 - L_1 - L_2)$$

$$\frac{\partial \psi_i}{\partial L_1} = \frac{\partial \psi_i}{\partial x}\frac{\partial x}{\partial L_1} + \frac{\partial \psi_i}{\partial y}\frac{\partial y}{\partial L_1}$$

$$\frac{\partial \psi_i}{\partial L_2} = \frac{\partial \psi_i}{\partial x}\frac{\partial x}{\partial L_2} + \frac{\partial \psi_i}{\partial y}\frac{\partial y}{\partial L_2}$$

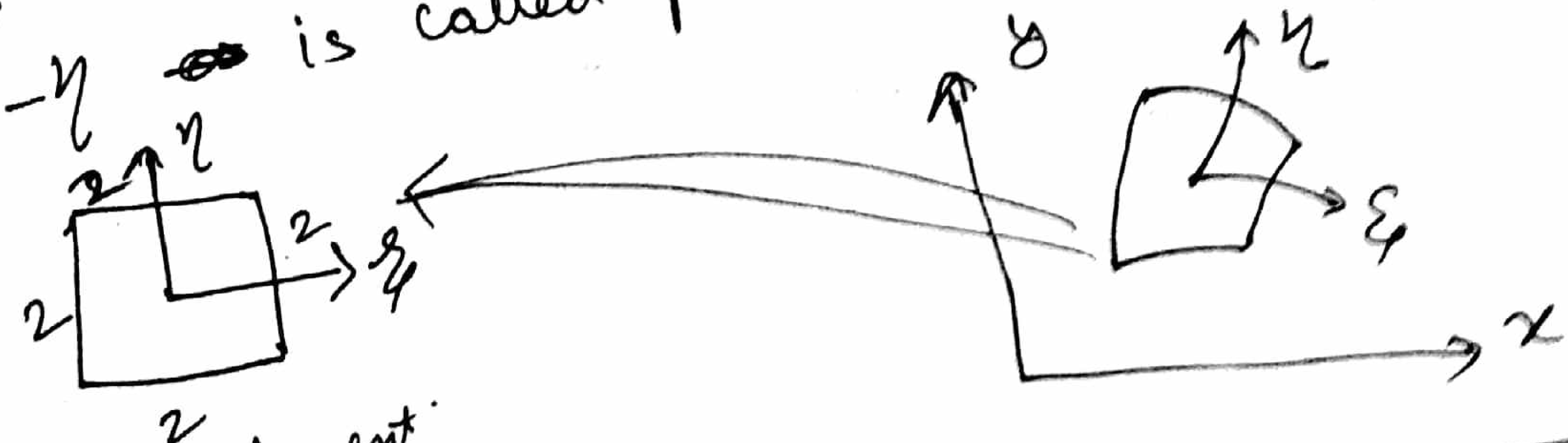
$$\begin{Bmatrix} \frac{\partial \psi_i}{\partial L_1} \\ \frac{\partial \psi_i}{\partial L_2} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial x}{\partial L_1} & \frac{\partial y}{\partial L_1} \\ \frac{\partial x}{\partial L_2} & \frac{\partial y}{\partial L_2} \end{Bmatrix} \begin{Bmatrix} \frac{\partial \psi_i}{\partial x} \\ \frac{\partial \psi_i}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial \psi_i}{\partial x} \\ \frac{\partial \psi_i}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial L_1} & \frac{\partial y}{\partial L_1} \\ \frac{\partial x}{\partial L_2} & \frac{\partial y}{\partial L_2} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\partial \psi_i}{\partial L_1} \\ \frac{\partial \psi_i}{\partial L_2} \end{Bmatrix}$$



Notes

$\xi - \eta$ is called parent element.



Parent element.

Obtused angle normally should not be more than 160°

Acute angle should be more than 20°.

Aspect ration in rectangle element should be b/w 1:1 than 1:20

Time dependent

$u(x,t) = U(x,t) = \sum_{j=1}^{n} u_j(t)\,\psi_j(x)$

$u(x,t)$ ↓ analytical or actual solution; $U(x,t)$ ↓ approximate solⁿ

$u(x,t) = T(t)\,X(s)$

① spatial approximation
- solving time problem at particular time step.

② **Time approximation** → How you relate the solution at given time step and next time step. ~~$\{u\}_s\ \{u\}_{s+1}$~~

$\{u\}_s \xrightarrow[\text{time step}]{\text{small}} \{u\}_{s+1}$

$b = EI$

Weak form

$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2 u}{\partial x^2}\right) + c_0 u + c_1\frac{\partial u}{\partial t} + c_2\frac{\partial^2 u}{\partial t^2} = f(x,t)$

subjected B.C.S

EBC. ① $u(x,t)$ or $-a\frac{\partial u}{\partial x}(x,t) + \frac{\partial}{\partial x}\left(b\frac{\partial^2 u}{\partial x^2}\right)$

(ii) $\frac{\partial u}{\partial x}(x,t)$ or $b\frac{\partial^2 u}{\partial x^2}$

(iii) initial condition $c_2 u(x,0)$ and $c_2\dot{u}(x,0) + c_1\dot{u}(x,0)$

Semi-discrete FE formulation

$\int_\Omega w\left[-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2 u}{\partial x^2}\right) + c_0 u + c_1\frac{\partial u}{\partial t} + c_2\frac{\partial^2 u}{\partial t^2} - f\right]d\Omega = 0$

differentiated twice

$\Rightarrow \int_\Omega \left(a\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + b\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 u}{\partial x^2} + c_0 wu + c_1 w\frac{\partial u}{\partial t} + c_2 w\frac{\partial^2 u}{\partial t^2} - wf\right)d\Omega - \hat{Q}_2\left(-\frac{\partial w}{\partial x}\right)\Big|_{x_A} - \hat{Q}_4\left(-\frac{\partial w}{\partial x}\right)\Big|_{x_B} = 0$

$-\hat{Q}_1\, w(x_A) - \hat{Q}_3\, w(x_B)$

$\hat{Q}_1 = \left[-a\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}b\frac{\partial^2 u}{\partial x^2}\right]_{x_A}, \quad \hat{Q}_2 = \left(b\frac{\partial^2 u}{\partial x^2}\right)\Big|_{x_A}$

$$\hat{Q}_3 = -\left[-a\frac{\partial u}{\partial x} + \frac{\partial}{\partial x} b\left(\frac{\partial^2 u}{\partial x^2}\right)\right]_{x_B}, \quad \hat{Q}_4 = -\left(b\frac{\partial^2 u}{\partial x^2}\right)\Big|_{x_B}$$

$$u(x,t) = \sum_{j=1}^{n} u_j(t)\,\psi_i(x)$$

$$0 = \int_\Omega \Big\{ a\frac{d\psi_i}{dx} \sum_{j=1}^{n} \frac{d\psi_j}{dx} u_j + b\frac{\partial^2\psi_i}{\partial x^2}\left(\sum_{j=1}^{n}\left(\frac{\partial^2\psi_j}{\partial x^2} u_j\right)\right) + c_1\psi_i\left(\sum_{j=1}^{n}\psi_j(x)\,\dot{u}_j(t)\right)$$
$$+ c_0\psi_i\left(\sum_{j=1}^{n}\psi_j u_j\right) + c_2\psi_i\left(\sum_{j=1}^{n}\psi_j(x)\,\ddot{u}_j(t)\right) - \psi_i f \Big\}\, d\Omega$$
$$- \hat{Q}_i$$

$$0 = \sum_{j=1}^{n}\left[\left(k^1_{ij} + k^2_{ij} + M^0_{ij}\right) u_j + M^1_{ij}\frac{du_j}{dt} + M^2_{ij}\frac{\partial^2 u_j}{\partial t^2} - f_i\right]$$

$$M^1_{ij} = \int c_1 \psi_i \psi_j\, dx$$

$$M^0_{ij} = \int c_0 \psi_i \psi_j\, dx \qquad k^1_{ij} = \int a\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\, dx$$

$$M^2_{ij} = \int c_2 \psi_i \psi_j\, dx$$

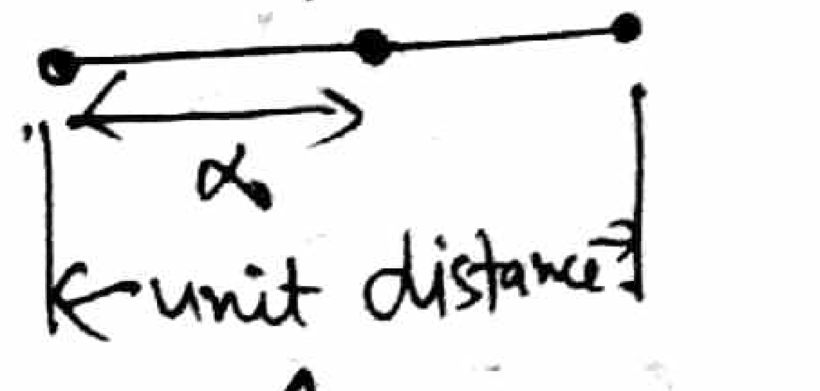
$$k^2_{ij} = \int b\frac{\partial^2\psi_i}{\partial x^2}\frac{\partial^2\psi_j}{\partial x^2}\, dx \qquad f_i = \int \psi_i f\, dx + \hat{Q}_i$$

$$[M][\dot{u}] + [K][u] = \{f\} \qquad \text{(Parabolic)}$$

↑ Mass matrix

Time approximation

$$\dot{u} = \frac{u_2 - u_1}{t_2 - t_1} \quad \{\text{from FDM}\}$$



$$\dot{u}_{s,s+1} = \frac{u_{s+1} - u_s}{\Delta t}$$

$\Delta t \rightarrow$ time step assummed to be equal

$$(y_2 - y) = \frac{y_2 - y_1}{x_2 - x_1} * (x_2 - x)$$

$$\dot{u}_{s+1} - \dot{u}_{s,s+1} = \frac{(\dot{u}_{s+1} - \dot{u}_s)}{1} (1-\alpha)$$

$$\boxed{\dot{u}_{s,s+1} = \alpha \dot{u}_{s+1} + (1-\alpha) \dot{u}_s}$$

$$\boxed{\alpha \dot{u}_{s+1} + (1-\alpha) \dot{u}_s = \frac{u_{s+1} - u_s}{\Delta t}} \quad -①$$

α family of approximation

$$-②$$

$$[M]\{\dot{u}\}_s + [k]_s \{u\}_s = \{f\}_s \quad -③$$

$$[M]\{\dot{u}\}_{s+1} + [k]_{s+1}\{u\}_{s+1} = \{f\}_{s+1}$$

Now premultiplying eq ① by $[M] \Delta t$

$$\Delta t [M] \alpha \dot{u}_{s+1} + \Delta t [M] (1-\alpha) \dot{u}_s = [M]\{u\}_{s+1} - [M]\{u\}_s \quad -④$$

From ③, ② & ④

$$\Delta t \alpha \left\{ \{f\}_{s+1} - [k]_{s+1} \{u\}_{s+1} \right\} + \Delta t (1-\alpha) \left\{ \{f\}_s - [k]_s \{u\}_s \right\}$$

$$= [M]\{u\}_{s+1} - [M]\{u\}_s$$

$$\Rightarrow [M]\{u\}_{s+1} + \Delta t \alpha [k]_{s+1} \{u\}_{s+1} = [M]\{u\}_s - \Delta t (1-\alpha)[k]_s \{u\}_s + \Delta t (1-\alpha)\{F\}_s + \Delta t \alpha \{F\}_{s+1}$$

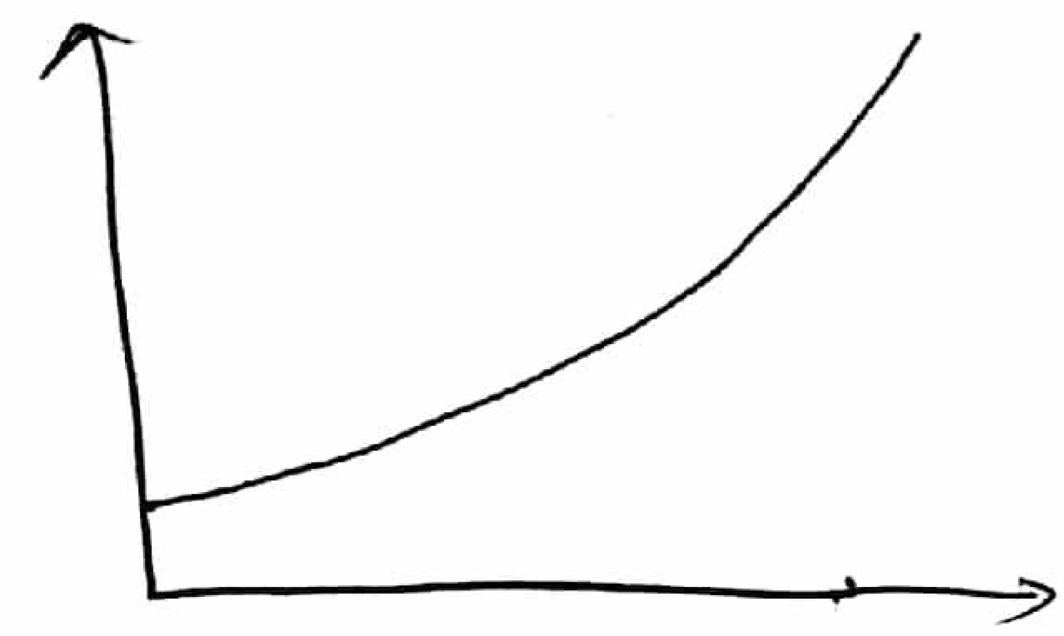
$$\Rightarrow [\hat{k}]\{u\}_{s+1} = [\bar{k}]\{u\}_s + \{\hat{F}\}_{s,s+1}$$

Algorithm

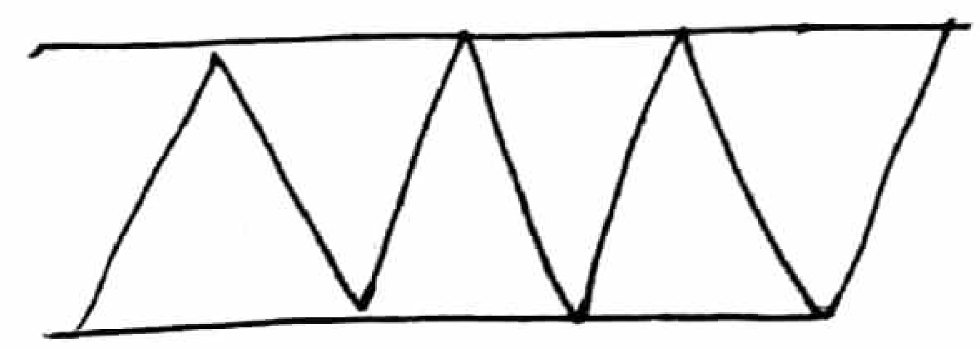
$$([M] + \Delta t \alpha [K]) \{u\}_{s+1} = ([M] - \Delta t (1-\alpha) [K]) \{u\}_s + \Delta t (\alpha \{F\}_{s+1} + (1-\alpha) \{F\}_s)$$

$\{u\}_s$ = sol^n from initial cond^n.

Unstable sol^n



Bounded



$\alpha = 0 \rightarrow$ Forward difference scheme

$\alpha = 1 \rightarrow$ Backward

$\alpha = \frac{1}{2} \rightarrow$ Crank - Nicholas

$\alpha = \frac{2}{3} \rightarrow$ Galerkin.

If $\alpha = 0$, the above eq^n reduces to where $\{f\}$ includes terms only at time s.

$$[M] \{u\}_{s+1} = \{f\}$$

$$\{u\}_{s+1} = [M]^{-1} \{f\}$$

$$\int_0^h C \psi_i \psi_j \, dx = \frac{Ch}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \text{consistent Mass matrix}$$

$$\rightarrow \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix} \leftarrow \text{lumped matrix.}$$

Adv:- You need not do matrix inversion.

There are two methods for making lumped matrix

① Row - sum lumping

$$M_{11} = \sum_{j=1}^{N} \int_0^h C \psi_i \psi_j \, dx = \int_0^h C \psi_i \, dx = \frac{Ch}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Delta t < t_{cr} = \frac{2}{(1-2\alpha)\lambda}$, for all $\alpha < 1/2$

↑ timestep

$\lambda \rightarrow$ Max^m eigen value of the problem

If $\Delta t > t_{cr}$ → The problem will be unbounded.

When $\alpha = 0$, it is called explicit scheme.

when $\alpha \neq 0$, it is called implicit scheme.

$\lambda =$ max^m eigenvalue of the k matrix

~~If $\alpha = \frac{1}{2}$~~

If $\alpha = \frac{1}{2}$, $\Delta t = ?$ → assess Δt.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad 0 < x < 1$$

B.C. $u(0,t) = 0$. $\frac{\partial u}{\partial t}(1,t) = 0$

I.C $u(0,0) = 0$. $u(1,0) = 1$

$\Rightarrow u_{1}(s) = 0$ $\qquad \Rightarrow u_2(s) = 1$

Let's consider $\alpha = \frac{1}{2}$

Mass matrix → consistent mass matrix

$$\int_0^h w\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) dx = \int_0^h \psi_i \psi_j \{\dot{u}\} dx - \int_0^h \frac{\partial^2 u}{\partial x^2} w \, dx$$

on integrating we get $[M]\{\dot{u}\} + [K][u] = [Q]$

$$\int_0^h w \frac{\partial u}{\partial t} dx = \int_0^h \psi_i \frac{d}{dt}\left(\sum u_j \psi_j\right) dx = \int_0^h \psi_i \psi_j \{\dot{u}\} dx$$

we get

$$\frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

One element model

$$\begin{bmatrix} \frac{h}{3}+\alpha\frac{\Delta t}{h} & \frac{h}{6}-\frac{\alpha\Delta t}{h} \\ \frac{h}{6}-\alpha\Delta t & \frac{h}{3}+\frac{\alpha\Delta t}{h} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{s+1} = ([M] + \alpha\Delta t [K]) \{u\}_{s+1}$$

$$([M] - (1-\alpha)\Delta t [K]) = \begin{bmatrix} \frac{h}{3} - (1-\alpha)\frac{\Delta t}{h} & \frac{h}{6} + \frac{(1-\alpha)\Delta t}{h} \\ \frac{h}{6} + \frac{(1-\alpha)\Delta t}{h} & \frac{h}{3} - \frac{(1-\alpha)\Delta t}{h} \end{bmatrix}$$

If $\{F\}$ does not change with time then
$\alpha\{F\}_{s+1} + \{F\}_s(1-\alpha) = \{F\}_s \Rightarrow \{F\}_s = \{Q\}$

Given $\frac{\partial u}{\partial t}(1,0) = 0 \Rightarrow Q_2 = 0$

$$\begin{bmatrix} \frac{h}{3}+\frac{\alpha\Delta t}{h} & \frac{h}{6}-\frac{\alpha\Delta t}{h} \\ \frac{h}{6}-\frac{\alpha\Delta t}{h} & \frac{h}{3}+\frac{\alpha\Delta t}{h} \end{bmatrix} \begin{bmatrix} \overset{0}{u_1} \\ u_2 \end{bmatrix}_{s+1} = \begin{bmatrix} \frac{h}{3}-(1-\alpha)\frac{\Delta t}{h} & \frac{h}{6}+\frac{(1-\alpha)\Delta t}{h} \\ \frac{h}{6}+\frac{(1-\alpha)\Delta t}{h} & \frac{h}{3}-\frac{(1-\alpha)\Delta t}{h} \end{bmatrix} \begin{bmatrix} \overset{0}{u_1} \\ u_2 \end{bmatrix}_s + \begin{bmatrix} Q_1 \\ \overset{0}{Q_2} \end{bmatrix}$$

$$\left(\frac{h}{3} + \frac{\alpha\Delta t}{h}\right)(u_2)_{s+1} = \frac{h}{3} - \frac{(1-\alpha)\Delta t}{h}(u_2)_s$$

if $\Delta t = 0.05$, $\alpha = 0.5$

Do using One quadratic element,

Eigen value problems

A & B are differential operators.

$A(u) = \lambda B(u)$

$-\frac{d^2u}{dx^2} = \lambda u$

DE: $\rho c A \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(kA \frac{\partial u}{\partial x}\right) = q(x,t) \to 0$ → homogeneous form

$u(x,t) = U(x)\, e^{-\lambda t}$ → sol^n (from variable separation method)

Parabolic eq^n type.

$$-\frac{d}{dx}\left(kA \frac{dU(x)}{dx}\right) - \lambda \rho c A\, U(x) = 0$$

← Governing eq^n for FEM sol^n.

↑ Eigen value problem after substituting the sol^n

On substituting the sol^n in DE.

$$\int_0^h w\left(-\frac{d}{dx}\left(kA \frac{dU(x)}{dx}\right) - \lambda \rho c A U(x)\right) dx = 0$$

weighted integral form

Q.2 $\rho A \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x}\right) = q(x,t)$ ← hyperbolic eqⁿ

solⁿ → $u(x,t) = U(x)\, e^{-i\omega t}$ $i = \sqrt{-1}$

Substitute the solⁿ in Differential eqⁿ.

$$\left[\rho A\, U(x)(+i^2\omega^2) - \frac{d}{dx} EA \frac{dU}{dx}\right] e^{-i\omega t} = 0$$

$$-\rho A\, U(x)\,\omega^2 - \frac{d}{dx} EA \frac{dU(x)}{dx} = 0$$

ω = natural frequency.

domain $0 \le x \le 1$

$$-\frac{d^2 u}{dx^2} + \frac{\partial u}{\partial t} = 0$$

$u(0,t) = 0$; $\left(\frac{\partial u}{\partial x} + u\right)\Big|_{x=1} = 0$

$u(x,t) = U(x)\, e^{-\lambda t}$.

$u(0,t) = U(0)\, e^{-\lambda t} = 0 \Rightarrow \boxed{U(0) = 0}$

$\left(\frac{dU(x)}{dx} + U(x)\right)\Big|_{x=1} = 0 \Rightarrow \frac{dU(x)}{dx}\Big|_{x=1} + U(1) = 0$

$Q_2^2 = -U_3 = Q_3$

Substitute $u(x,t) = U(x)e^{-\lambda t}$ in DE

$$-\frac{d^2 U}{dx^2} - \lambda U = 0$$

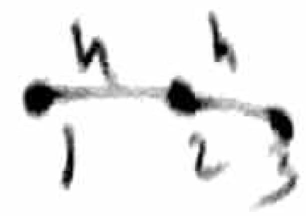
$$\int_0^h w\left(-\frac{d^2 U}{dx^2} - \lambda U\right) dx = 0$$

$$\int_0^h \frac{\partial w}{\partial x}\frac{\partial U}{\partial x}\, dx - \int_0^h w \lambda U\, dx - w\frac{dU}{dx} = 0$$

$$\int_0^h \left(\frac{d\psi_i}{dx}\frac{d}{dx}\sum_{j=1}^{n} \psi_j U_j\right) dx - \lambda \int_0^h \psi_i \left(\sum_{j=1}^{n} U_j \psi_j\right) dx = Q_i$$

$$[k_{ij}]\{U_j\} - \lambda[M_{ij}]\{U_j\} = \{Q_i\}$$

Considering linear elements (minimum elements are two)



$$\left(\frac{1}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} - \frac{\lambda h}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{bmatrix}U_1\\ U_2\end{bmatrix} = \begin{bmatrix}Q_1\\ Q_2\end{bmatrix}$$

$$\left\{\frac{1}{h}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix} - \frac{\lambda h}{6}\begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}\right\}\begin{Bmatrix}U_1 = 0\\ U_2\\ U_3\end{Bmatrix} = \begin{Bmatrix}Q_1^1\\ Q_2^1 + Q_1^2\\ Q_2^2 = -U_3\end{Bmatrix}$$

Condensed matrix directly for U_2 and U_3

$$\begin{bmatrix}\frac{2}{h} - \frac{4\lambda h}{6} & -\frac{1}{h} - \frac{\lambda h}{6}\\ -\frac{1}{h} - \frac{\lambda h}{6} & \frac{1}{h} - \frac{2\lambda h}{6}\end{bmatrix}\begin{bmatrix}U_2\\ U_3\end{bmatrix} = \begin{bmatrix}0\\ -U_3\end{bmatrix}$$

determinant of the matrix below gives a 2nd order eqn of λ is called characteristic equation.

$$\begin{bmatrix}\frac{2}{h} - \frac{4\lambda h}{6} & -\frac{1}{h} - \frac{\lambda h}{6}\\ -\frac{1}{h} - \frac{\lambda h}{6} & \frac{1}{h} + 1 - \frac{2\lambda h}{6}\end{bmatrix}\begin{bmatrix}U_2\\ U_3\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}(a_{11} - b_{11}\lambda) & (a_{12} - b_{12}\lambda)\\ (a_{21} - b_{21}\lambda) & (a_{22} - b_{22}\lambda)\end{bmatrix}\begin{bmatrix}U_2\\ U_3\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

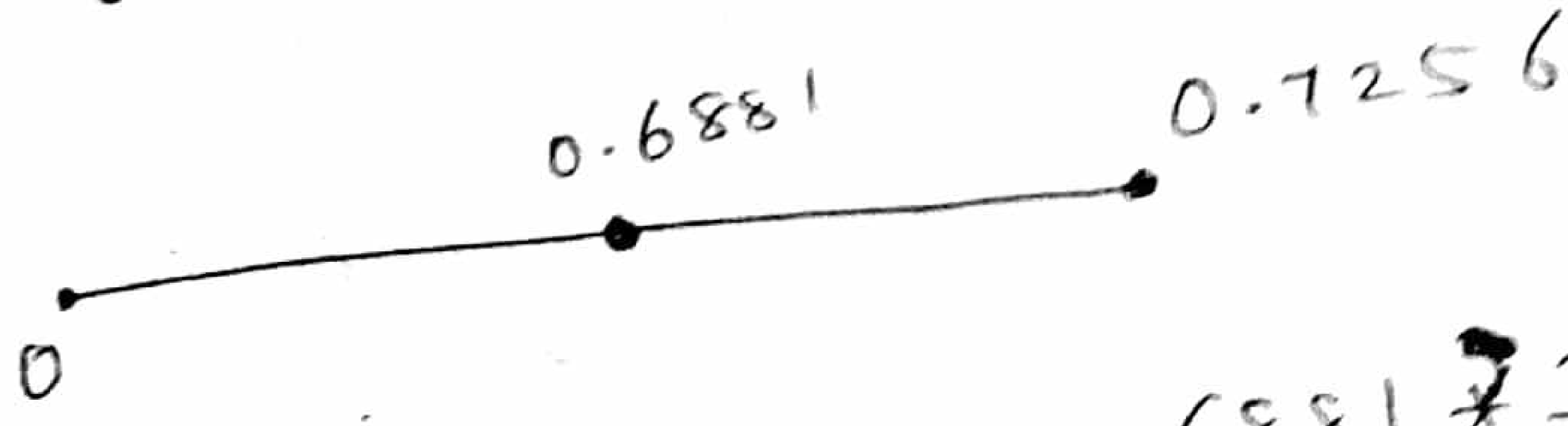
For $\lambda = \lambda_1$ (1st eigen value)

$$(a_{11} - b_{11}\lambda_1)U_2^{(1)} + (a_{12} - b_{12}\lambda_1)U_3^{(1)} = 0$$

$2.5033\, U_2 - 2.3742\, U_3 = 0$

$1.0\, U_2 - 1.0544\, U_3 = 0$

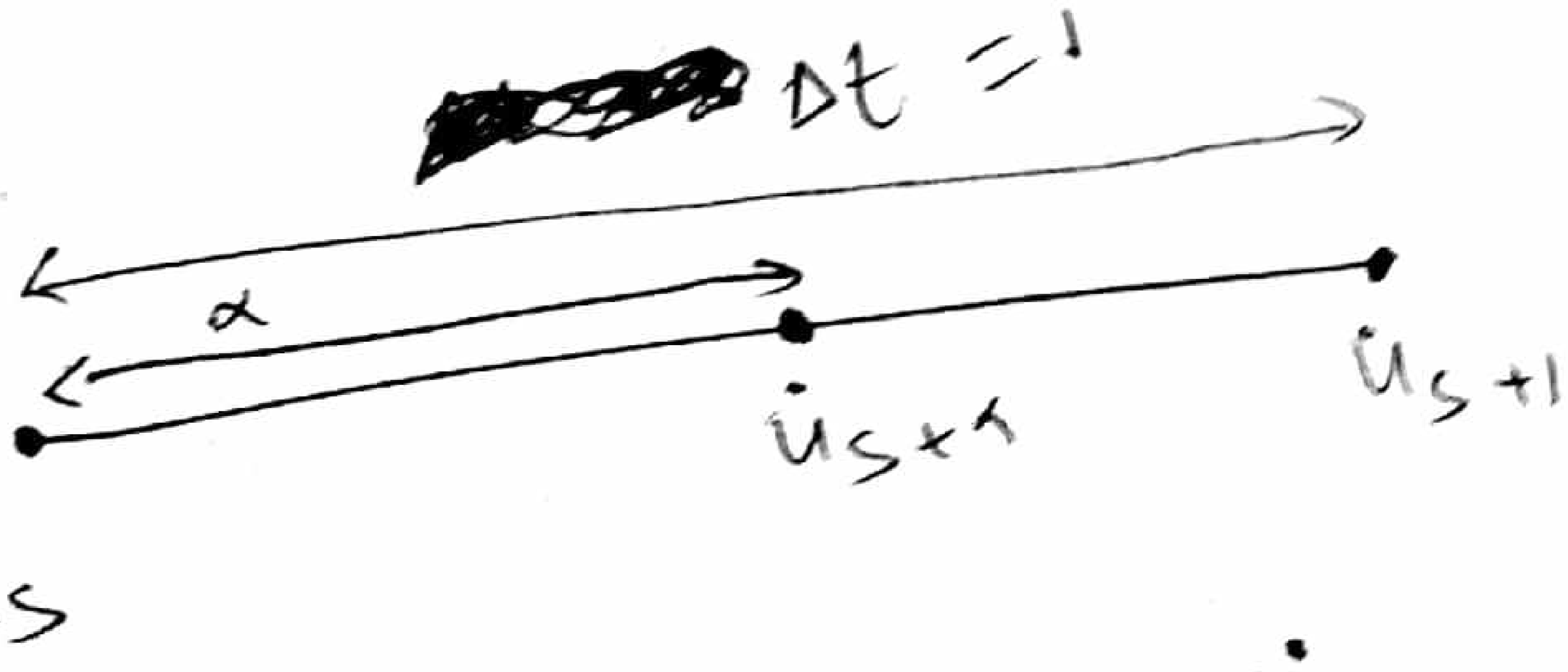
$0.6881\, U_2 - 0.7256\, U_3 = 0$



$$U'(x) = 0\left(1 - \frac{x}{h}\right) + 0.6881 \frac{x}{h}$$

$$U'(x) = 0.6881\left(1 - \frac{x}{h}\right) + 0.7256 \frac{x}{h}$$

$$\frac{du}{dt} \approx \frac{u_{s+1} - u_s}{\Delta t} = \dot{u}_{s+1} - \dot{u}_s$$

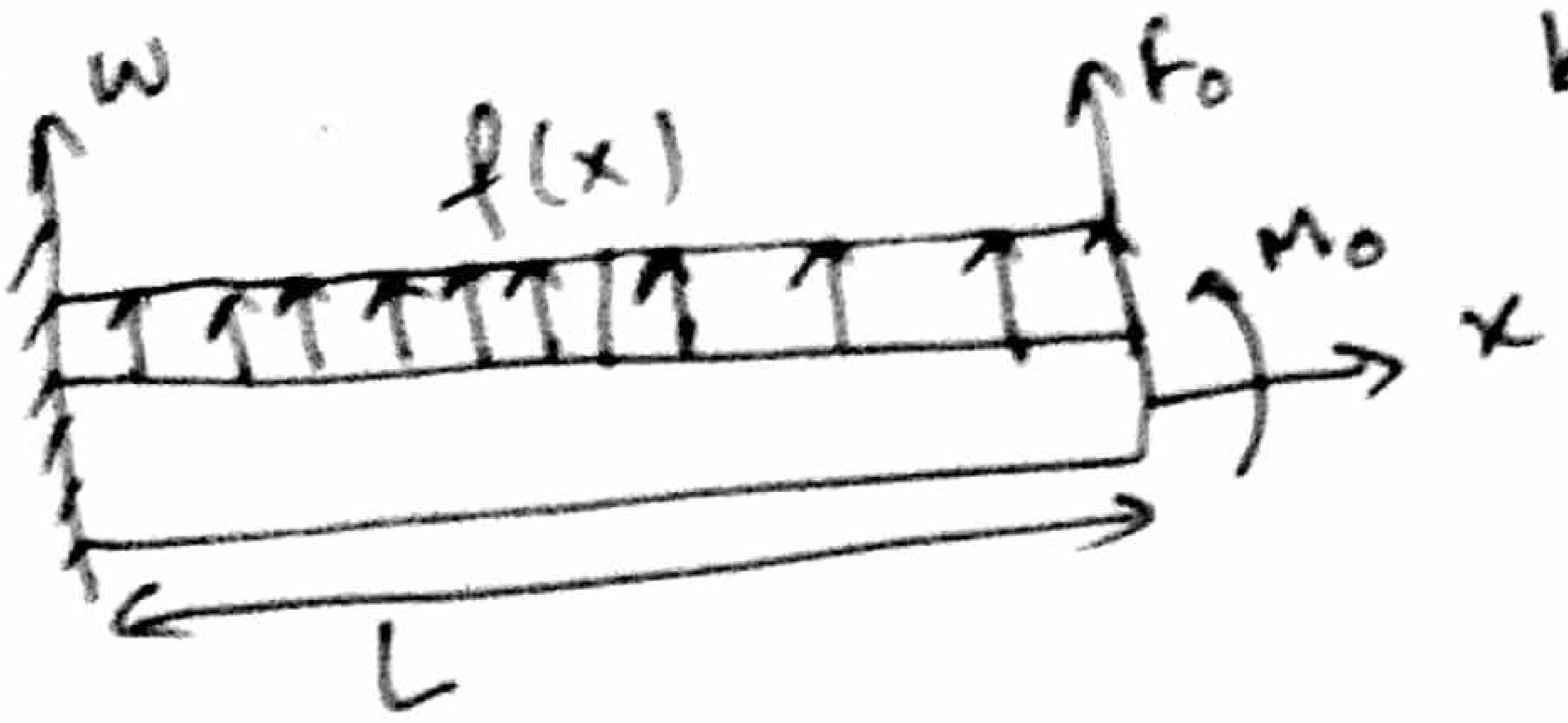


$$\frac{\dot{u}_{s+\alpha} - \dot{u}_s}{(\alpha - 0)} = \frac{\dot{u}_{s+1} - \dot{u}_s}{(1 - 0)}$$

$$\dot{u}_{s+\alpha} = \dot{u}_s(1-\alpha) + \dot{u}_{s+1}\alpha$$

$$= \dot{u}_s(1-\alpha) + \dot{u}_{s+1}\alpha$$

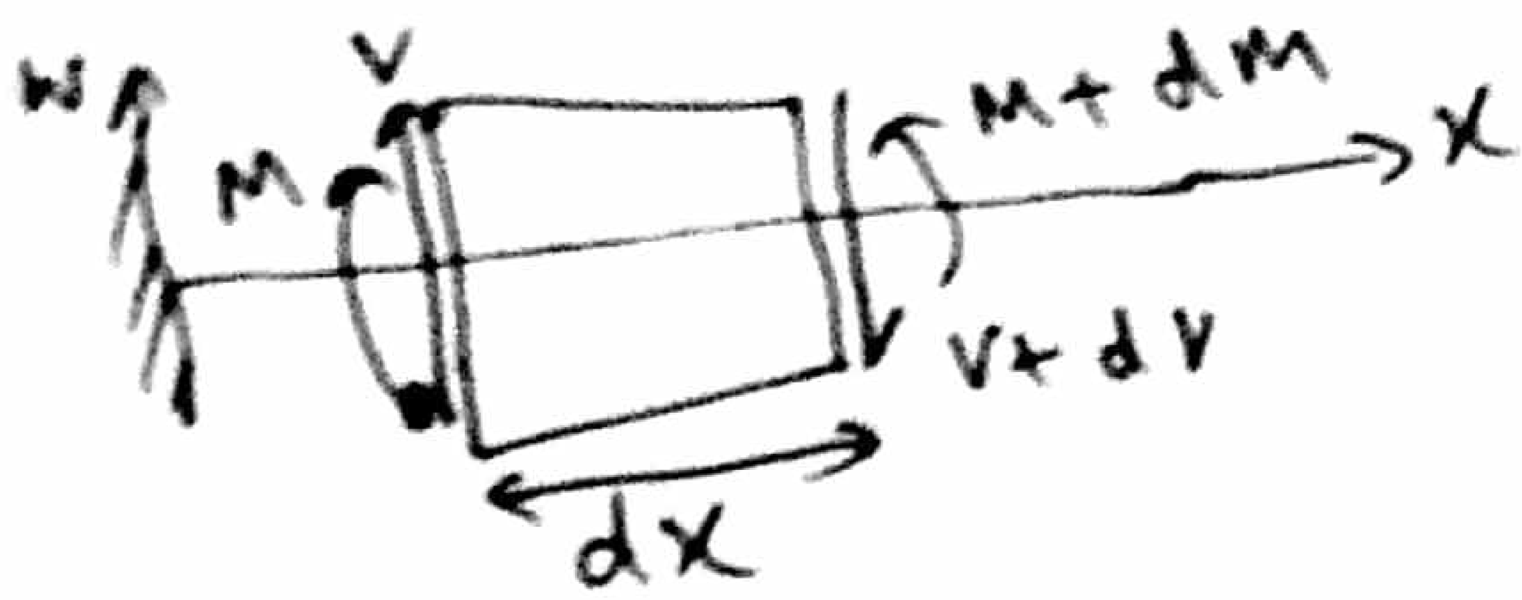
$$\frac{u_{s+\alpha} - u_s}{\Delta t} =$$



$b = EI$

$M = b\frac{d^2w}{dx^2}$

$V = b\frac{dM}{dx}$

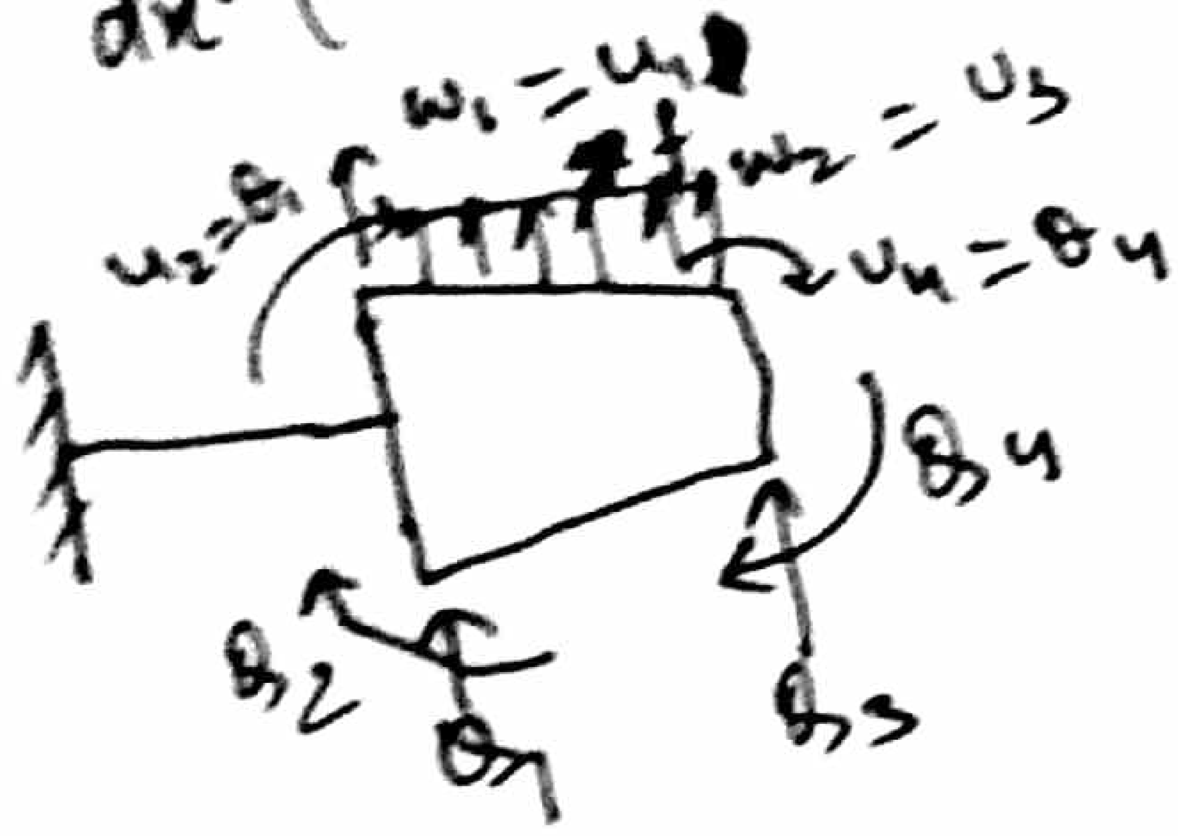


$\frac{dV}{dx} = f(x)$

$\frac{d^2}{dx^2}\left(b\frac{d^2w}{dx^2}\right) = f(x)$

$\int_{x_e}^{x_{e+1}} v\left[\frac{d^2}{dx^2} b\frac{d^2w}{dx^2} - f\right]dx = 0$

$v \Rightarrow$ weight function



$\int_{x_e}^{x_{e+1}}\left[-\frac{dv}{dx}\frac{d}{dx}b\frac{d^2w}{dx^2} - vf\right]dx + \left[v\frac{d}{dx}b\frac{d^2w}{dx^2}\right]_{x_e}^{x_{e+1}} = 0$

θ_1, θ_4 are the slopes

Q_1, Q_3 are shear forces

Q_2 & Q_4 are the moments

2nd $\rightarrow$ $\int_{x_e}^{x_{e+1}}\left[b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} - vf\right]dx + \left[v\frac{d}{dx}b\frac{d^2w}{dx^2} - \frac{dv}{dx}b\frac{d^2w}{dx^2}\right]_e^{e+1} = 0$

Coefficient of functions of weight function gives natural boundary condition

Where is field varaiable

NBC

Shearforce $\rightarrow$ $\frac{d}{dx}b\frac{d^2w}{dx}$

Moment $\rightarrow$ $b\frac{d^2W}{dx^2}$

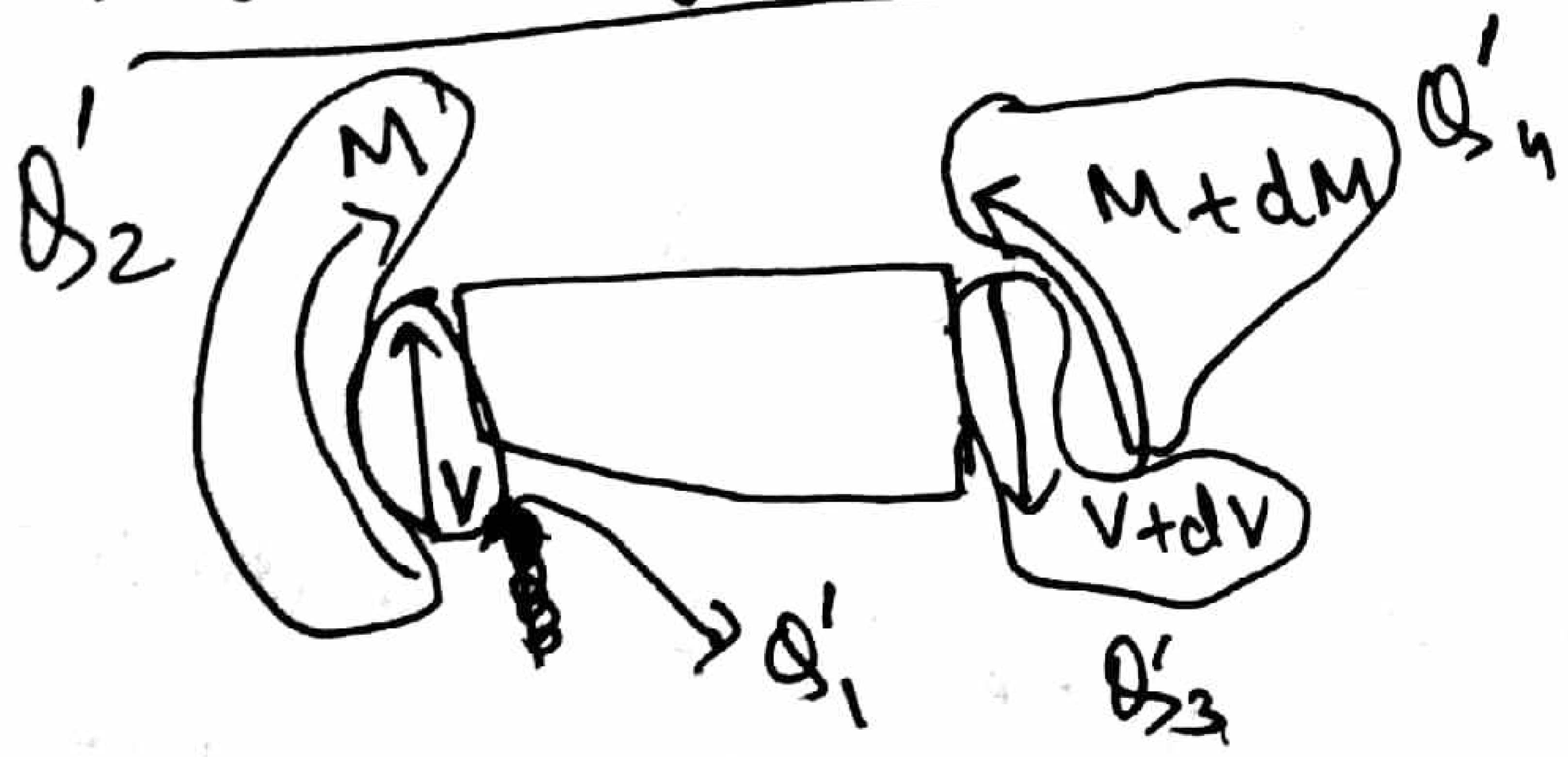
The weight f^n written in terms of field variable gives Essential boundary conditions

EBC: v, $\frac{dv}{dx}$

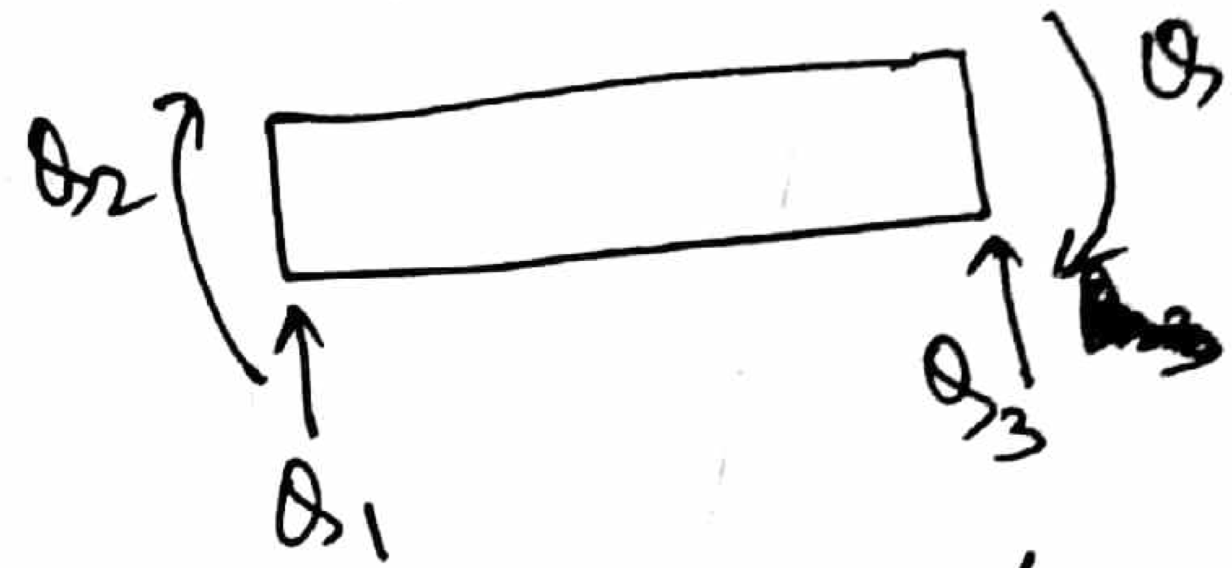
$$Q_1 = \left[\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_{x^e} \qquad Q_2 = \left(b\frac{d^2w}{dx^2}\right)_{x^e}$$

$$Q_3 = -\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)_{x^{e+1}} \qquad Q_4 = -\left(b\frac{d^2w}{dx^2}\right)_{x^{e+1}}$$

From strength of materials



For FEM



$$Q_3' = \frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)_{x^{e+1}}$$

$$Q_4' = \left(b\frac{d^2w}{dx^2}\right)_{x^{e+1}}$$

so, $Q_2 = -Q_3'$

$Q_4 = -Q_4'$

$$0 = \int_{x^e}^{x^{e+1}}\left(b\frac{d^2v}{dx^2}\frac{d^2w}{dx^2} - vf\right)dx - v(x^e)Q_1 - \left(-\frac{dv}{dx}\right)\Big|_{x^e} Q_2 - v(x^{e+1})Q_3 - \left(-\frac{dv}{dx}\right)\Big|_{x^{e+1}} Q_4$$

Weak form

Total no. of nodal variables at each node = 2 (two degrees of freedom per node)

Total no. of nodal variables = 4 (for one linear element)

Total no. of

u_1, θ_1 —— u_2, θ_2

$$w = c_1 + c_2x + c_3x^2 + c_4x^3$$

$$\theta = -\frac{dw}{dx}$$

$$u = w$$

$U_1 = w(x_e)$, $U_2 = -\left.\frac{dw}{dx}\right|_{x=x^e}$, $U_3 = w(x^{e+1})$, $U_4 = \left(-\frac{dw}{dx}\right)_{x^{e+1}}$

Generalised displacements u, θ represented by U.

$$U_1 = c_1 + c_2 x^e + c_3 (x^e)^2 + c_4 (x^e)^3$$

$$U_2 = 0 - c_2 - 2c_3 x^e - 3c_4 (x^e)^2$$

$$U_3 = c_1 + c_2 x^{e+1} + c_3 (x^{e+1})^2 + c_4 (x^{e+1})^3$$

$$U_4 = 0 - c_2 - 2c_3 x^{e+1} - 3c_4 (x^{e+1})^2$$

$$\{U\} = \begin{bmatrix} 1 & x^e & (x^e)^2 & (x^e)^3 \\ 0 & -1 & -2x^e & -3(x^e)^2 \\ 1 & x^{e+1} & (x^{e+1})^2 & (x^{e+1})^3 \\ 0 & -1 & -2x^{e+1} & -3x^{e+1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$$w(x) = U_1 \phi_1 + U_2 \phi_2 + U_3 \phi_3 + U_4 \phi_4$$

$$\phi_1 = 1 - 3\left(\frac{\bar{x}}{h}\right)^2 + 2\left(\frac{\bar{x}}{h}\right)^3$$

$$\phi_2 = -\bar{x}\left(1 - \frac{\bar{x}}{h}\right)^2$$

$$\phi_3 = 3\left(\frac{\bar{x}}{h}\right)^2 - 2\left(\frac{\bar{x}}{h}\right)^3$$

$$\phi_4 = -\bar{x}\left[\left(\frac{\bar{x}}{h}\right)^2 - \left(\frac{\bar{x}}{h}\right)\right]$$

Written in Local coordinates. No need to memorize, will be given in the exams

hermite cubic interpolation Functions

$$\frac{d\phi_1}{dx} = -\frac{6}{h}\frac{\bar{x}}{h}\left(1 - \frac{\bar{x}}{h}\right)$$

$$\frac{d\phi_2}{dx} = -\left[1 + 3\left(\frac{\bar{x}}{h}\right)^2 - 4\left(\frac{\bar{x}}{h}\right)\right]$$

$$\frac{d\phi_3}{dx} = -\frac{d\phi_1}{dx} \quad , \quad \frac{d\phi_4}{dx} = -\frac{\bar{x}}{h}\left[\frac{3\bar{x}}{h} - 2\right]$$

$U_1 = w(x_e)$, $U_2 = -\left.\frac{dw}{dx}\right|_{x=x^e}$, $U_3 = w(x^{e+1})$, $U_4 = \left(-\frac{dw}{dx}\right)_{x^{e+1}}$

Generalised displacements u, θ represented by U.

$$U_1 = c_1 + c_2 x^e + c_3 (x^e)^2 + c_4 (x^e)^3$$

$$U_2 = 0 - c_2 - 2c_3 x^e - 3c_4 (x^e)^2$$

$$U_3 = c_1 + c_2 x^{e+1} + c_3 (x^{e+1})^2 + c_4 (x^{e+1})^3$$

$$U_4 = 0 - c_2 - 2c_3 x^{e+1} - 3c_4 (x^{e+1})^2$$

$$\{U\} = \begin{bmatrix} 1 & x^e & (x^e)^2 & (x^e)^3 \\ 0 & -1 & -2x^e & -3(x^e)^2 \\ 1 & x^{e+1} & (x^{e+1})^2 & (x^{e+1})^3 \\ 0 & -1 & -2x^{e+1} & -3x^{e+1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$$w(x) = U_1 \phi_1 + U_2 \phi_2 + U_3 \phi_3 + U_4 \phi_4$$

{ Written in Local coordinates. No need to memorize, will be given in the exams

$$\phi_1 = 1 - 3\left(\frac{\bar{x}}{h}\right)^2 + 2\left(\frac{\bar{x}}{h}\right)^3$$

$$\phi_2 = -\bar{x}\left(1 - \frac{\bar{x}}{h}\right)^2$$

hermite cubic interpolation Functions

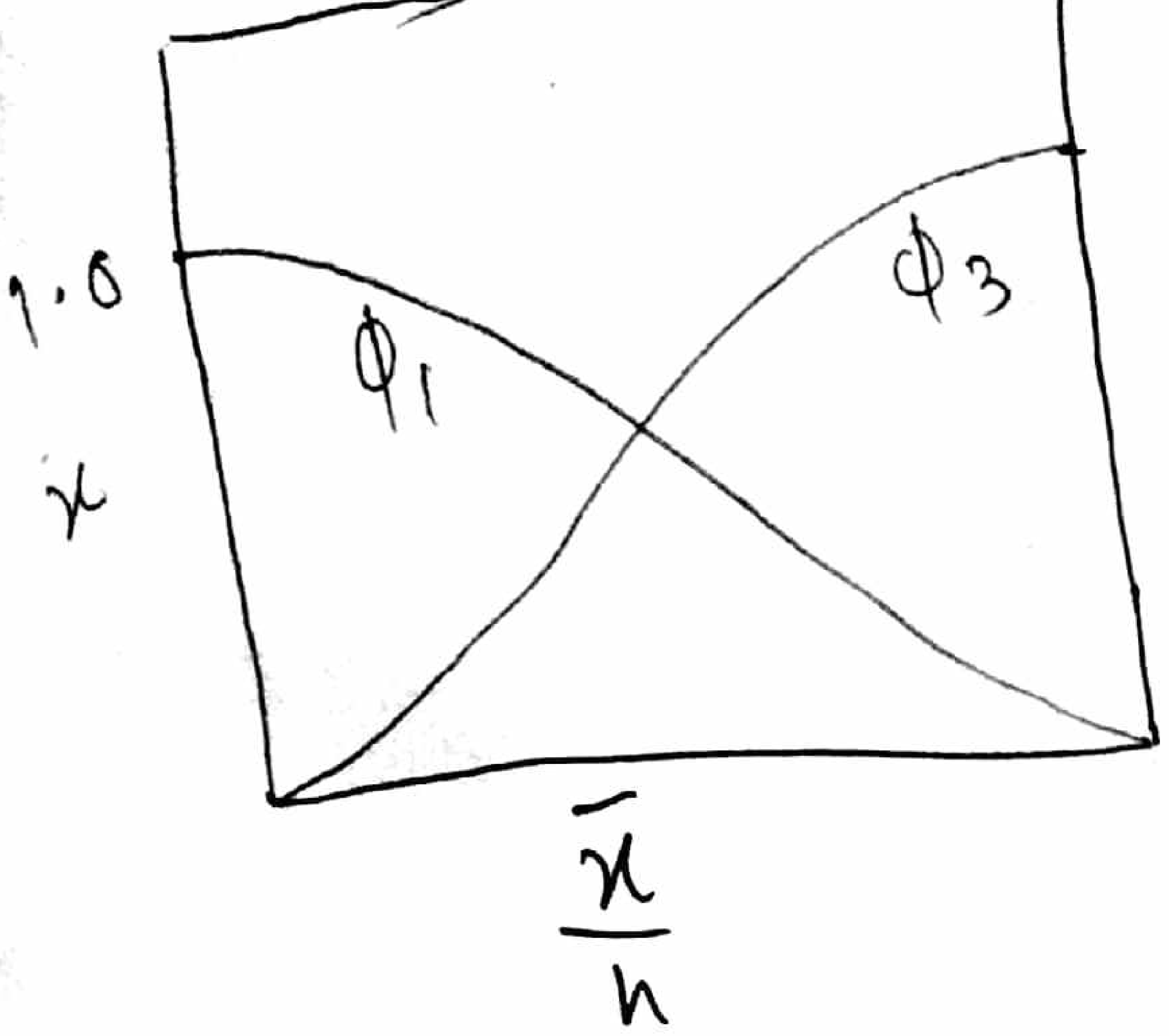
$$\phi_3 = 3\left(\frac{\bar{x}}{h}\right)^2 - 2\left(\frac{\bar{x}}{h}\right)^3$$

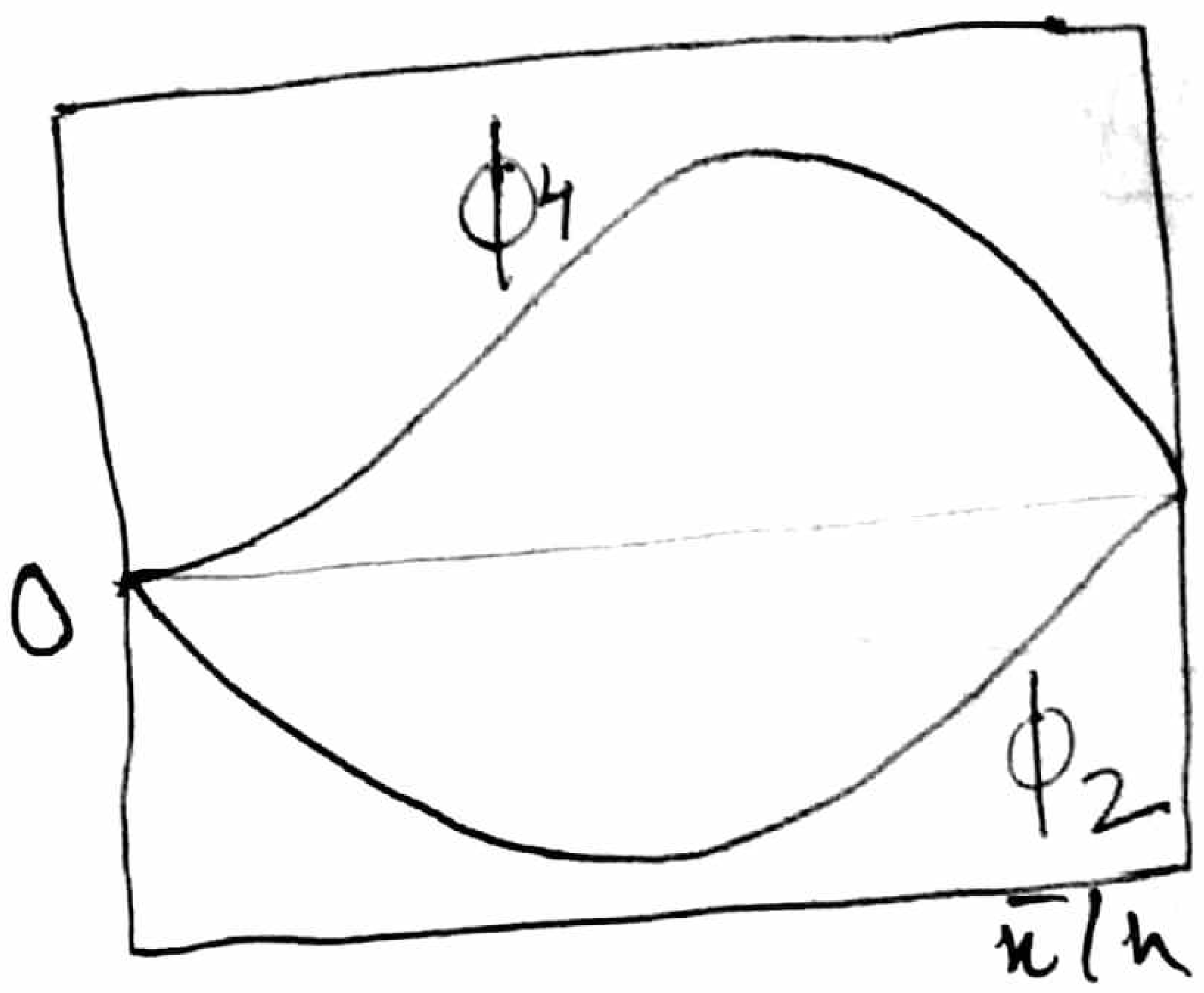
$$\phi_4 = -\bar{x}\left[\left(\frac{\bar{x}}{h}\right)^2 - \left(\frac{\bar{x}}{h}\right)\right]$$

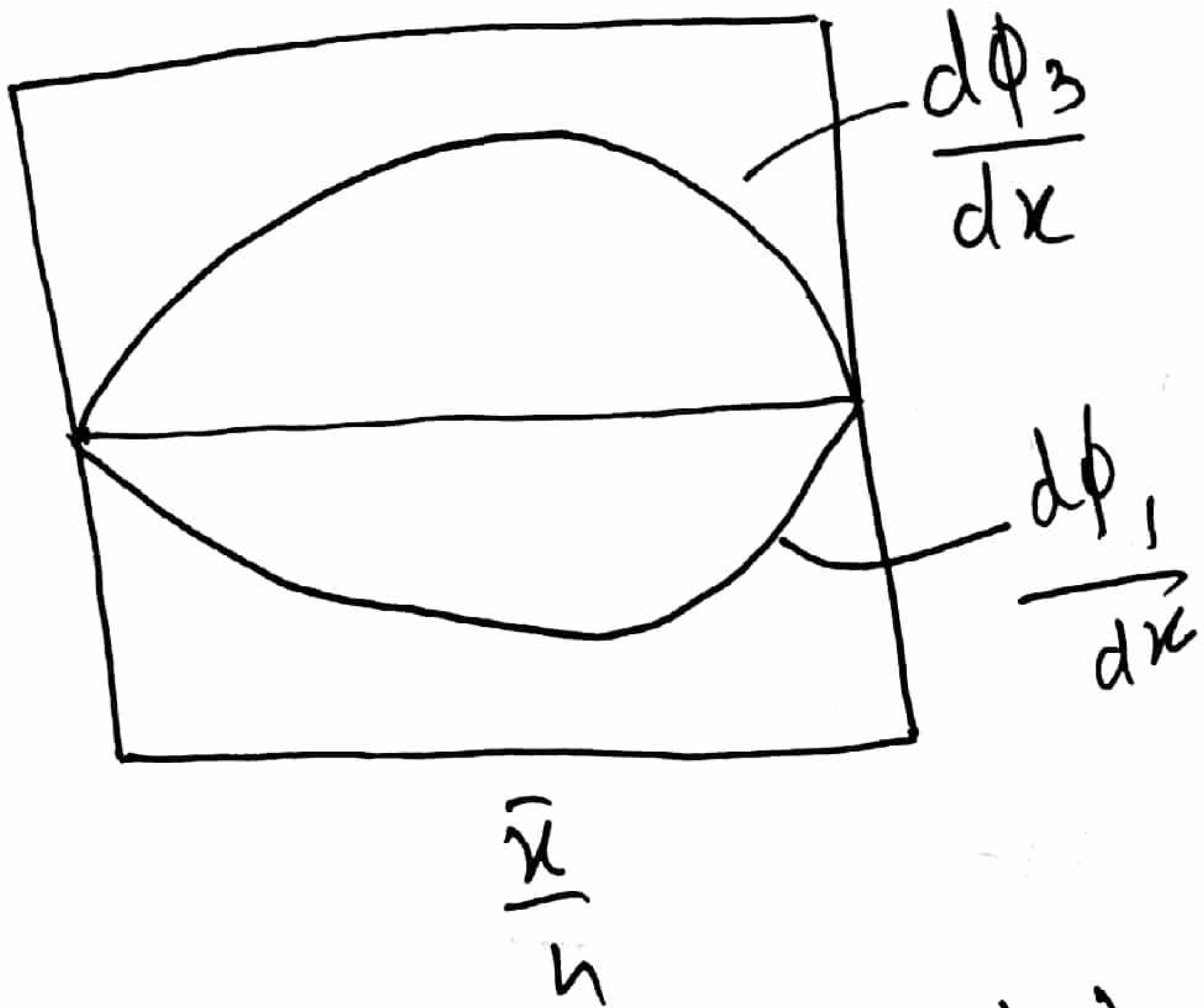
$$\frac{d\phi_1}{dx} = -\frac{6}{h}\frac{\bar{x}}{h}\left(1 - \frac{\bar{x}}{h}\right)$$

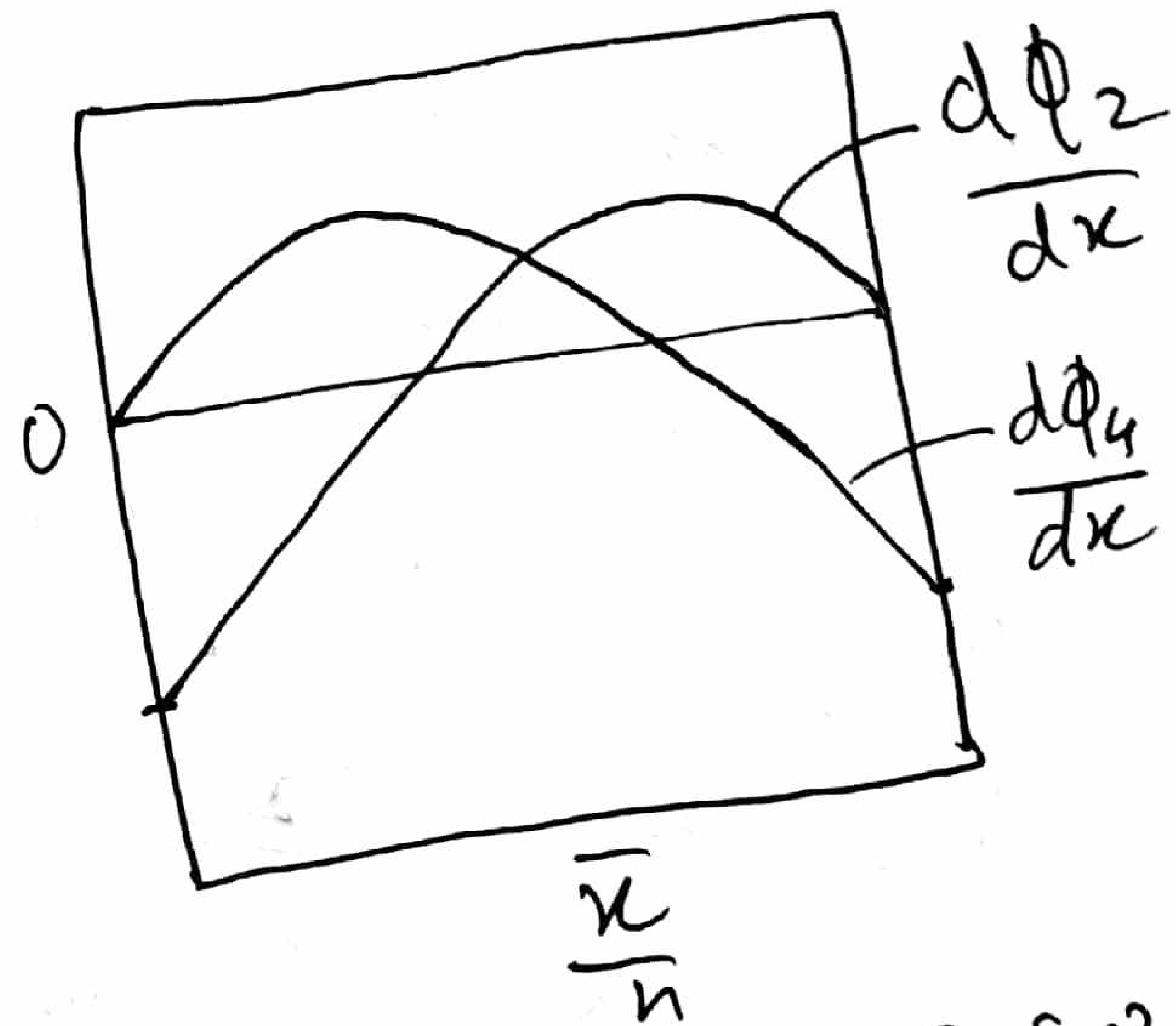
$$\frac{d\phi_2}{dx} = -\left[1 + 3\left(\frac{\bar{x}}{h}\right)^2 - 4\left(\frac{\bar{x}}{h}\right)\right]$$

$$\frac{d\phi_3}{dx} = -\frac{d\phi_1}{dx} \quad , \quad \frac{d\phi_4}{dx} = -\frac{\bar{x}}{h}\left[\frac{3\bar{x}}{h} - 2\right]$$









$$\int_0^h b \frac{d^2\phi_i}{dx^2} \sum_{j=1}^{4} \frac{d^2}{dx^2}(u_j \phi_j)\, dx - \int_0^h \phi_i f \, dx = \{Q\}$$

$$[k]\{u\} = \{f\} + \{Q\}$$

$$\{u\} = \begin{Bmatrix} u_1 \to \text{displacement} \\ u_2 \to \text{slope} \\ u_3 \to \text{displ.} \\ u_4 \to \text{slope} \end{Bmatrix}$$

$$[k] = \frac{2b}{h^3}\begin{bmatrix} 6 & -3h & -6 & -3h \\ -3h & 2h^2 & 3h & h^2 \\ -6 & 3h & 6 & 3h \\ -3h & h^2 & 3h & 2h^2 \end{bmatrix}$$

$$\{F\} = \frac{fh}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix} + \begin{Bmatrix} Q_1 \to \text{shear force} \\ Q_2 \to \text{bending moment} \\ Q_3 \to \text{shear force.} \\ Q_4 \to \text{bending moment} \end{Bmatrix}$$

$$\Rightarrow \{K\}\{u\} = \{F\}$$

Assembly

Two linear elements

For 1 - DOF

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

For one element

Assembly

$$\begin{bmatrix} k_{11} & k^1_{12} & 0 \\ k^1_{21} & k^1_{22}+k^2_{11} & k^2_{12} \\ 0 & k^2_{21} & k^2_{22} \end{bmatrix}$$

For 2 - DOF

$$\left[\begin{array}{cc|cc} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ \hline k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{array}\right] \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}$$

(blocks marked: (k_{11}), (k_{12}), (k_{21}), (k_{22}))

Kassembly

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & 0 & 0 \\ k_{21} & k_{22} & k_{23} & k_{24} & 0 & 0 \\ k_{31} & k_{32} & k^1_{33}+k^2_{11} & k^1_{34}+k^2_{12} & k_{13} & k_{14} \\ k_{41} & k_{42} & k^1_{43}+k^2_{21} & k^1_{44}+k^2_{22} & k_{23} & k_{24} \\ 0 & 0 & k_{31} & k_{32} & k_{33} & k_{34} \\ 0 & 0 & k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}$$

(k_{11})	(k_{12})	0
(k_{22})	$(k^1_{22}+k^2_{11})$	(k^2_{12})
0	(k^2_{21})	(k^2_{22})

Q

EI → frictional rigidity

~~[K]~~ , $U_1 = 0$, $U_2 = 0$

$$\frac{2b}{h^3}\begin{bmatrix}6 & -3h & -6 & -3h\\ -3h & 2h^2 & 3h & h^2\\ -6 & 3h & 6 & 3h\\ -3h & h^2 & 3h & 2h^2\end{bmatrix}\begin{bmatrix}U_1\\U_2\\U_3\\U_4\end{bmatrix}=\begin{bmatrix}Q_1\\Q_2\\Q_3\\Q_4\end{bmatrix}$$

$$\frac{2b}{h^3}\begin{bmatrix}6 & 3h\\ 3h & 2h^2\end{bmatrix}\begin{bmatrix}U_3\\U_4\end{bmatrix}=\begin{bmatrix}-P\\0\end{bmatrix}$$

$$\frac{2b}{h^3}(6U_3 + 3hU_4) = -P$$

$$(3hU_3 + 2h^2U_4) = 0 \Rightarrow U_4 = -\frac{3}{2h}U_3$$

$$\frac{2b}{h^3} \quad 6U_3 + 3h\left(\frac{-3}{2h}U_3\right) = -\frac{Ph^3}{2b}$$

$$\left(6 - \frac{9}{2}\right)U_3 = -\frac{Ph^3}{2b}$$

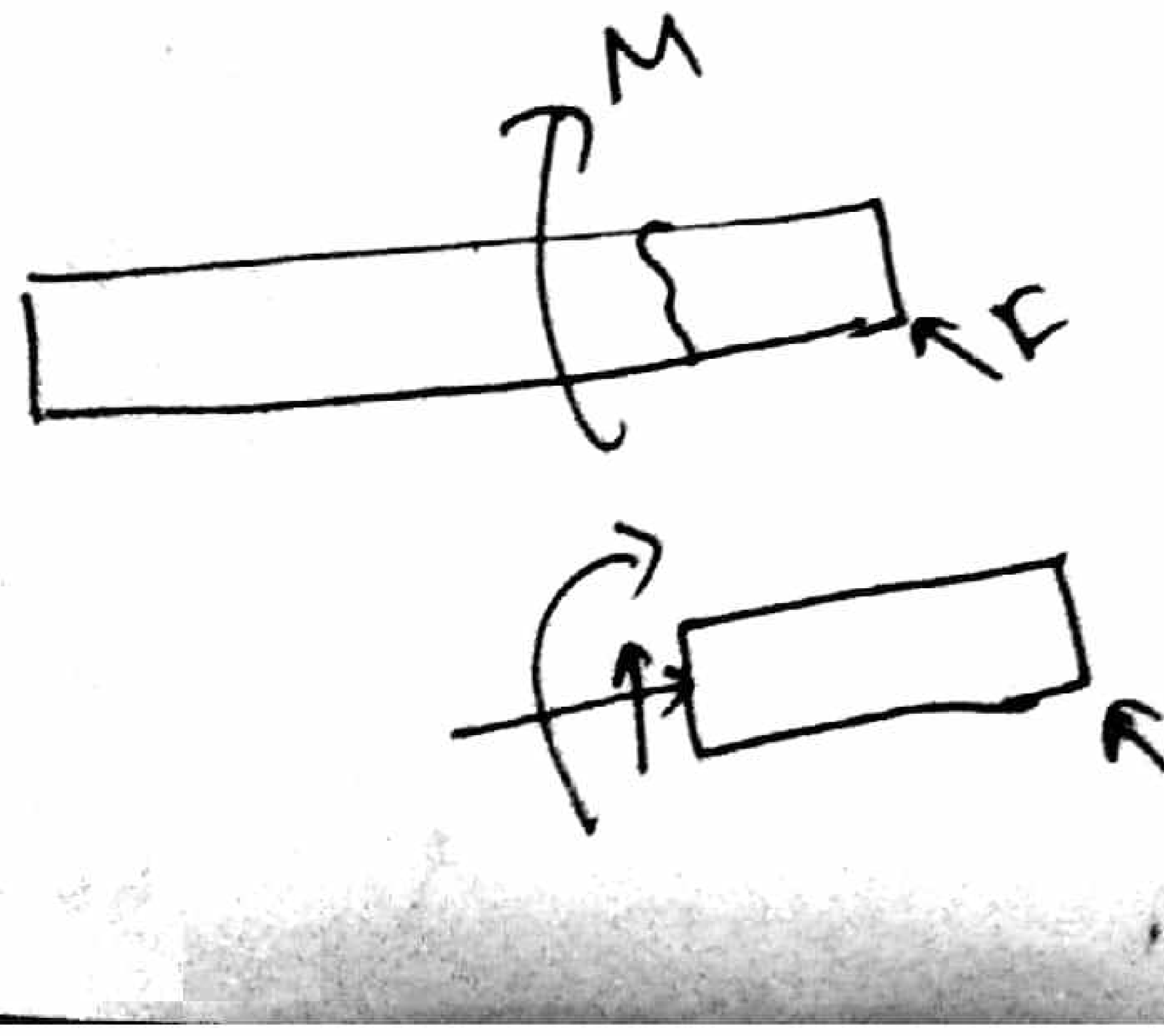
$$\boxed{U_3 = -\frac{Ph^3}{3b}}$$

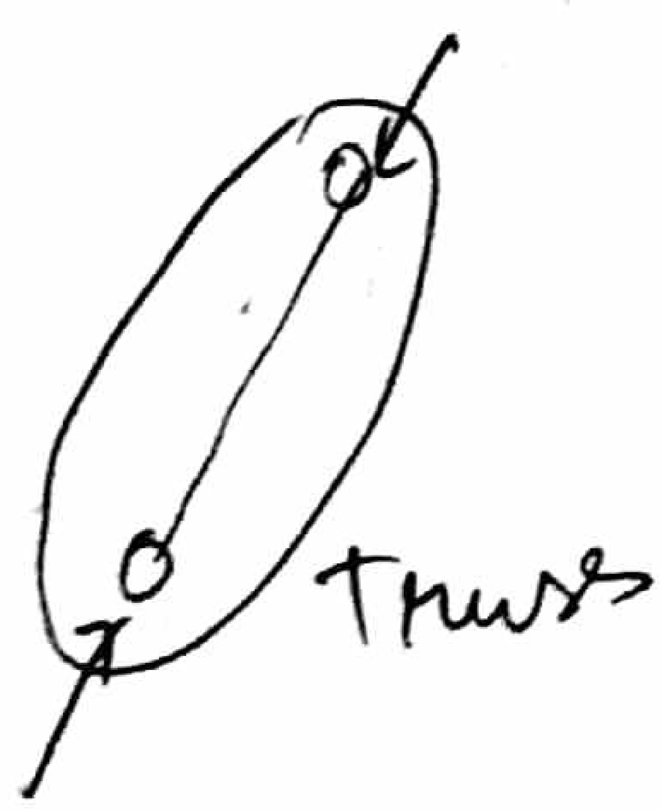
Post processing

Difference b/w truss and ~~structure~~ Frame

Truss is always a 2 force member.

Frame always has some moment.





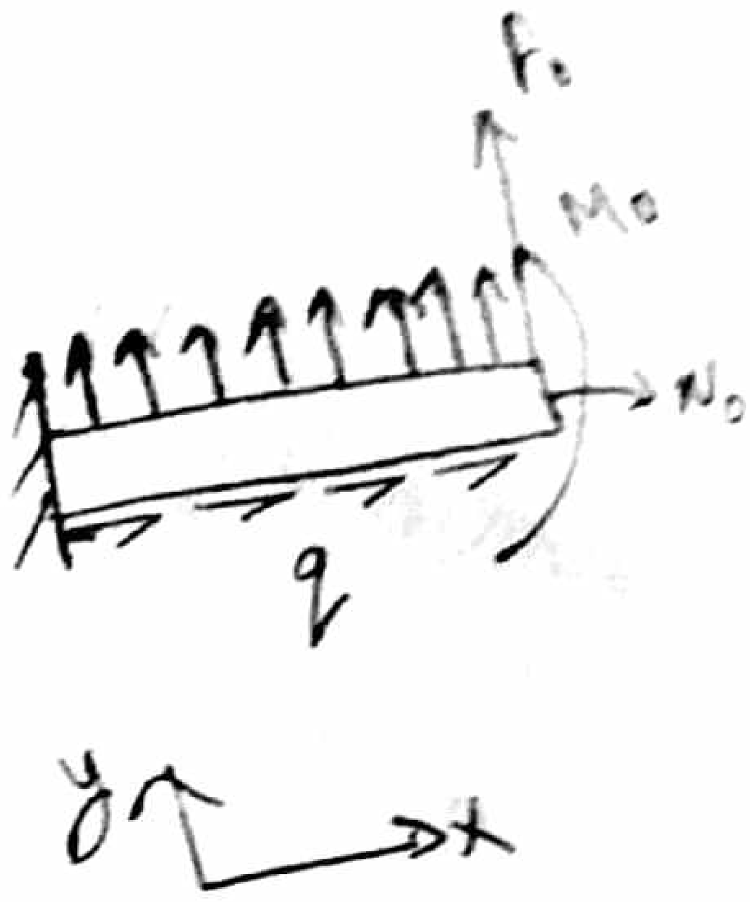
Generalised problem (generalised load)

$$[k]\{u\} = \frac{2EI}{h^3}\begin{bmatrix} \mu & 0 & 0 & -\mu & 0 & 0 \\ 0 & 6 & -3h & 0 & -6 & -3h \\ 0 & -3h & 2h^2 & 0 & 3h & h^2 \\ -\mu & 0 & 0 & \mu & 0 & 0 \\ 0 & -6 & 3h & 0 & 6 & 3h \\ 0 & -3h & h^2 & 0 & 3h & 2h^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$\mu = \frac{Ah^2}{2I}$$

These eq^{n}s are only for conditions where deflection are not very large.

Two nodal element.
1st & 4th are due to the axial loading



$$\{F\} = \begin{Bmatrix} \frac{1}{2}qh \leftarrow \text{axial} \\ \frac{1}{2}fh \\ -\frac{1}{12}fh^2 \\ \frac{1}{2}qh \leftarrow \text{axial} \\ \frac{1}{2}fh \\ \frac{1}{12}fh^2 \end{Bmatrix} + \begin{Bmatrix} Q_1 \leftarrow \text{axial} \\ Q_2 \\ Q_3 \\ Q_4 \leftarrow \text{axial} \\ Q_5 \\ Q_6 \end{Bmatrix}$$

For two elements

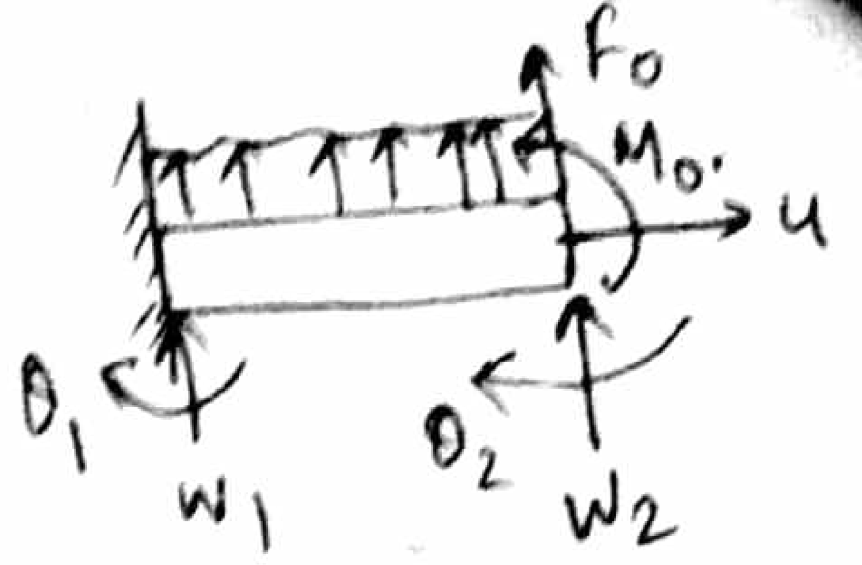
k_{11}^1	k_{12}^1
k_{21}^1	k_{22}^1

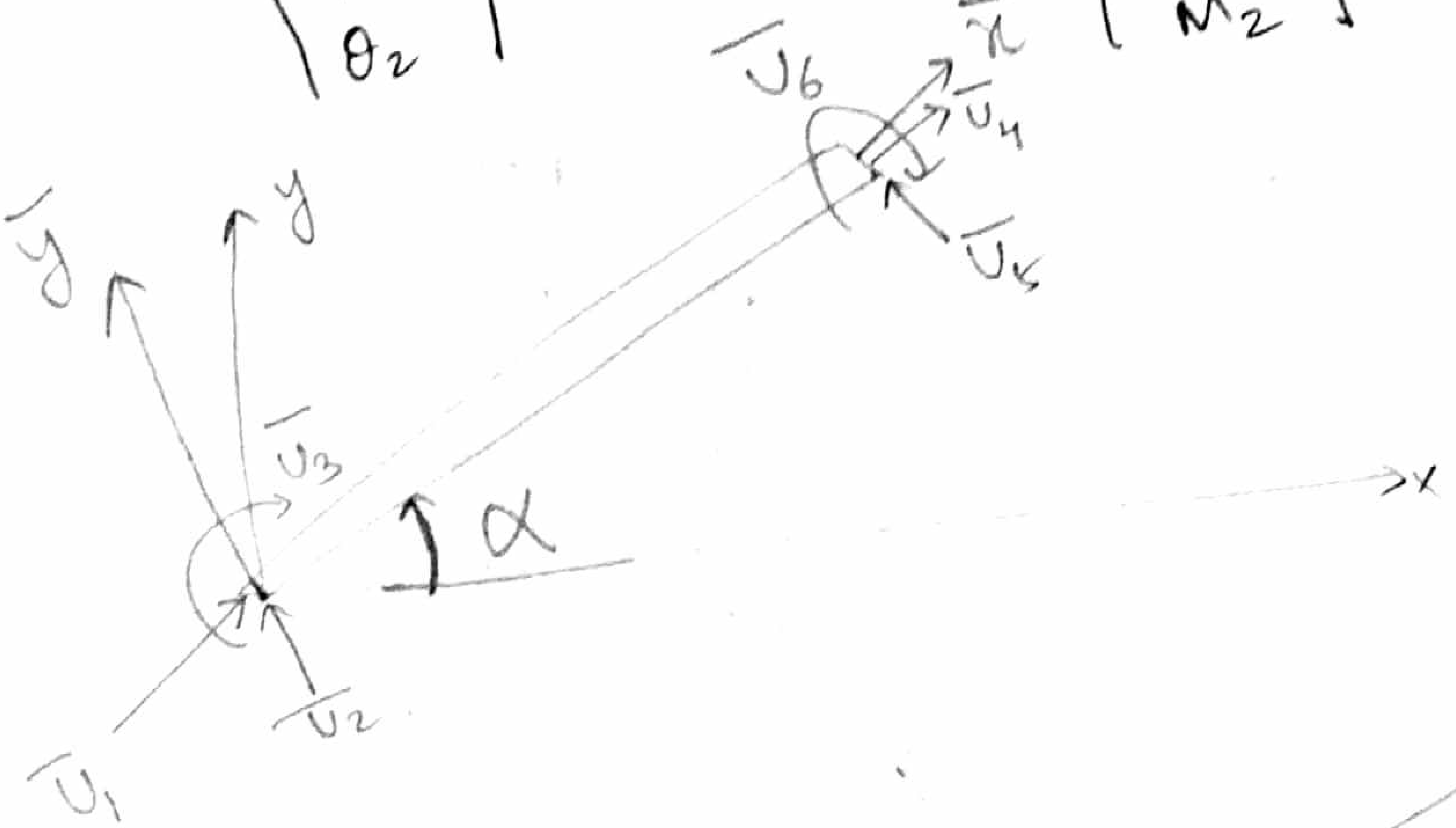
k_{11}^2	k_{12}^2
k_{21}^2	k_{22}^2

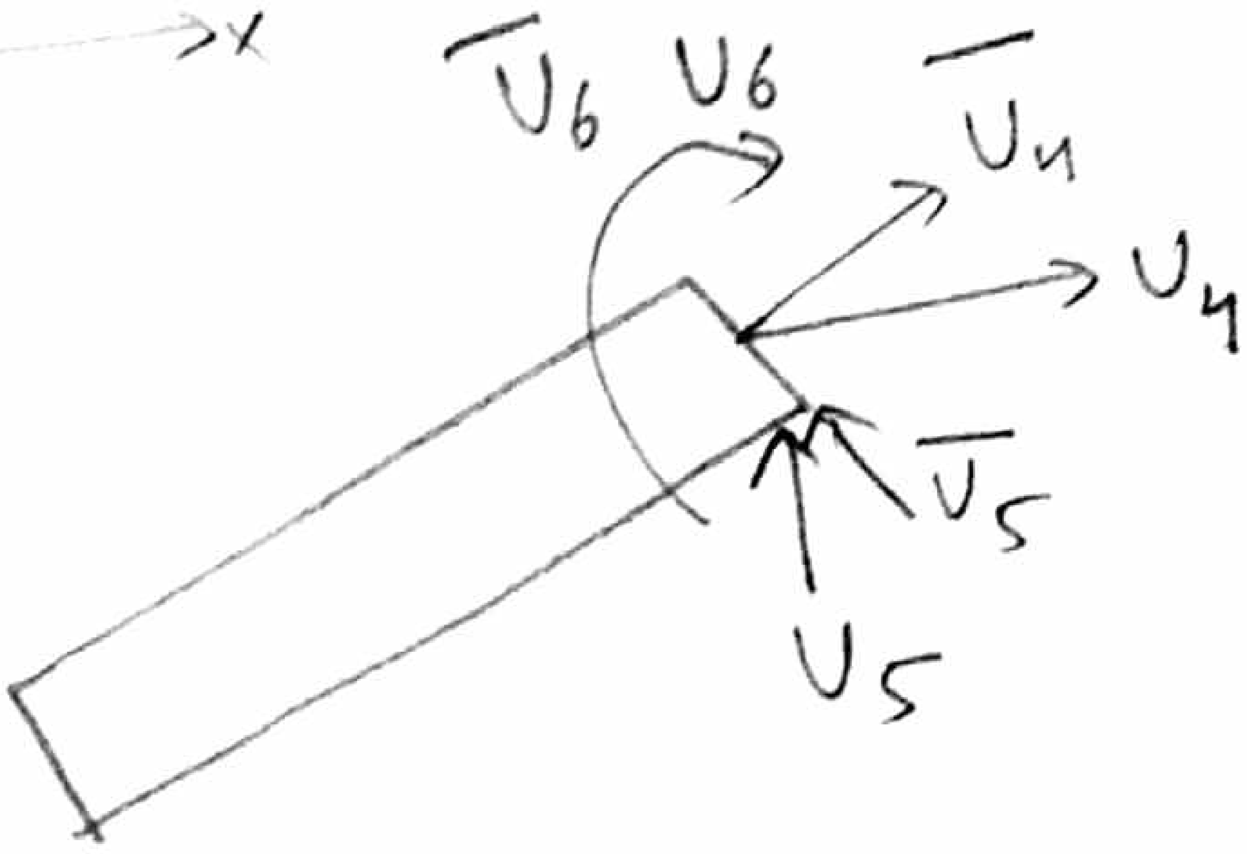
k_{11}^1	k_{12}^1	0
k_{21}^1	$k_{22}^1 + k_{11}^2$	k_{12}^2
0	k_{21}^2	k_{22}^2

$$\{u\} = \begin{Bmatrix} u_1 \to 0 \\ u_2 \to 0 \\ u_3 \to 0 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} \quad \{f\} = \begin{Bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 + f_1^2 \\ f_5^1 + f_2^2 \\ f_6^1 + f_3^2 \\ f_4^2 \\ f_5^2 \\ f_6^2 \end{Bmatrix} \quad Q = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 \\ Q_4^1 + Q_1^2 \to 0 \\ Q_5^1 + Q_2^2 \to 0 \\ Q_6^1 + Q_3^2 \to 0 \\ Q_4^2 \\ Q_5^2 \\ Q_6^2 \end{Bmatrix}$$

$$\{u\} = \begin{Bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{Bmatrix} \qquad \{Q\} = \begin{Bmatrix} N_1 \\ V_1 \\ M_1 \\ N_2 \\ V_2 \\ M_2 \end{Bmatrix}$$







$$\begin{bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix}$$

$\{\bar{u}\} = [T]\{u\}$ → displacements

$\{u\} = [T]^{-1}\{\bar{u}\} = [T]^T\{u\}$

$\{\bar{F}\} = [T]\{F\}$ → forces

$[\bar{K}][\bar{u}] = [\bar{F}]$

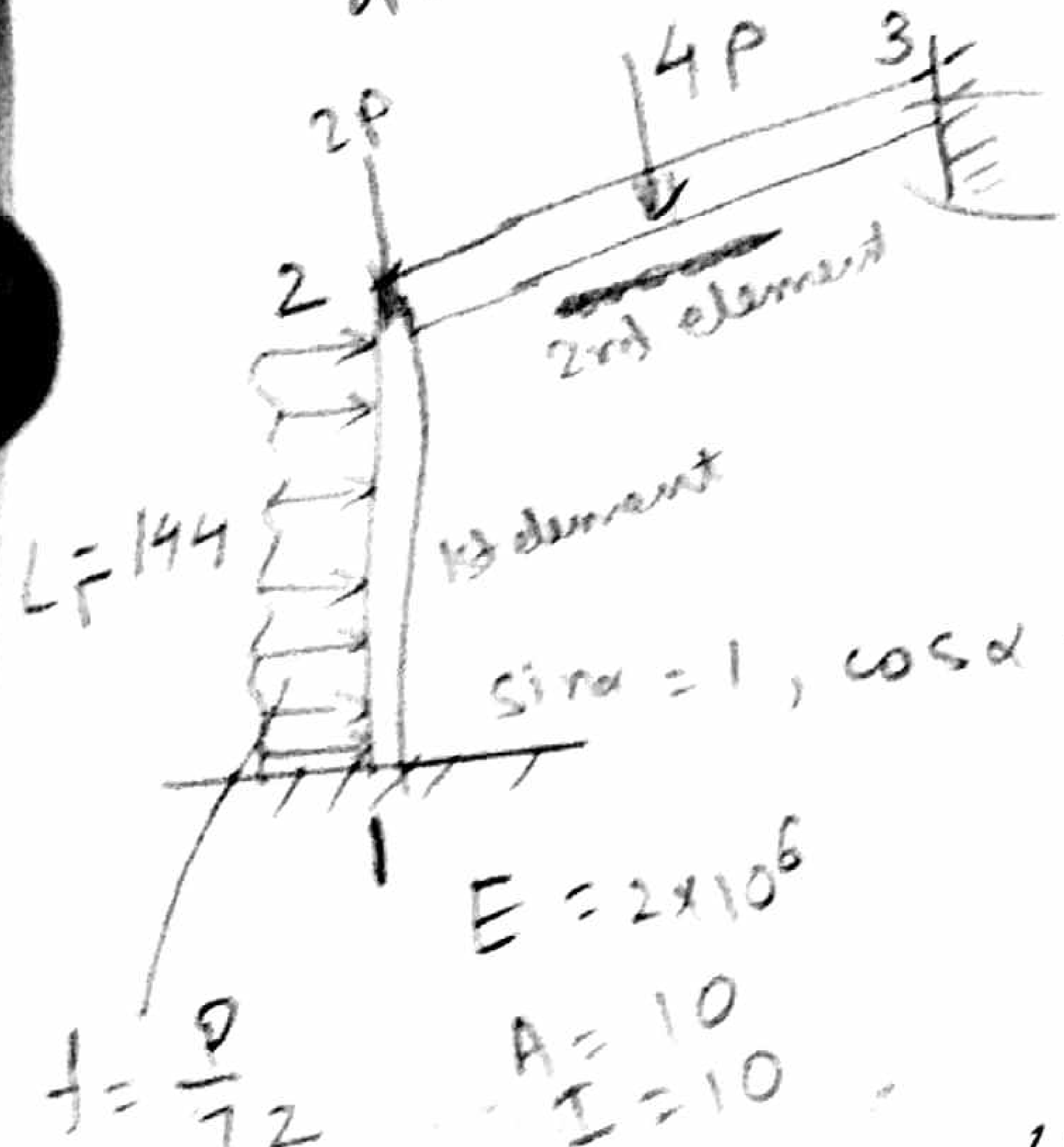
$[\bar{K}][T][u] = [T][F]$

$[T]^T[\bar{K}][T]\{u\} = [T]^{-1}[T]\{F\}$

$[K]\{u\} = \{F\}$

α is always measured from x axis in the counter clockwise direction.

Do not write α = [redacted] −60°



$L_2 = 180$

$\sin\alpha = 0.6$

$\cos\alpha = 0.8$

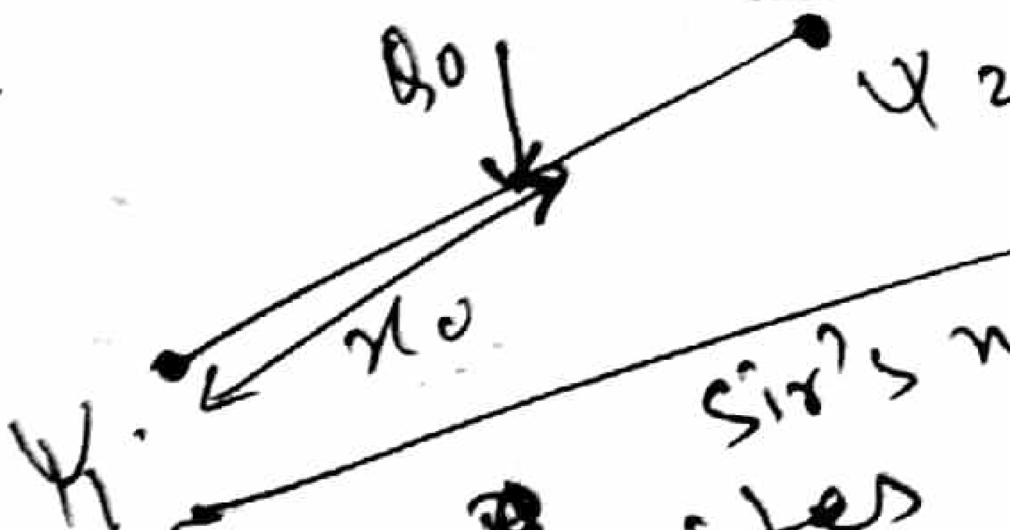
$q(x) = Q_0\,\delta(x - x_0)$

$\int_{-\infty}^{\infty} F(x)\,\delta(x - x_0)\,dx = f(x_0)$

$f_i = \int q(x)\,\psi_i\,dx$

$= \int Q_0\,\delta(x - x_0)\,\psi_i\,dx$

$= Q_0\,\psi_i(x_0)$



For axial load ← $\psi_1 = \left(1 - \frac{x}{h}\right)$

$\psi_1(x_0) = \left(1 - \frac{x_0}{h}\right)$

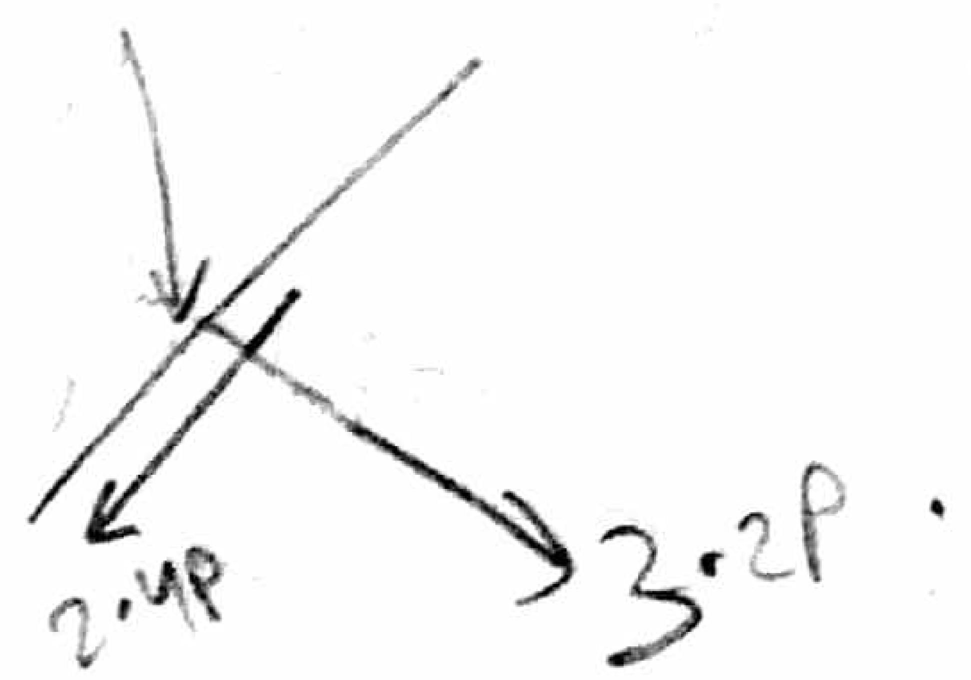
For axial load ← $\psi_2(x_0) = \frac{x_0}{h}$

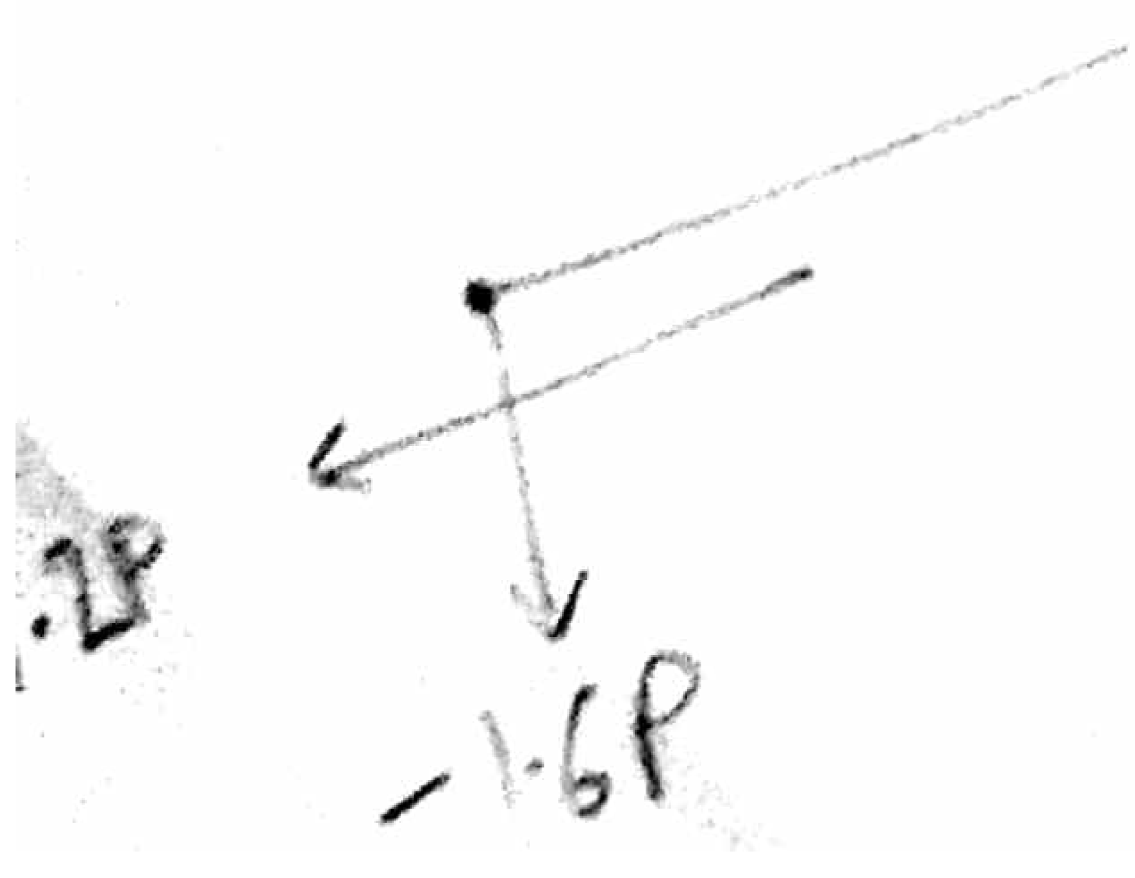
$(Q)_{\bullet} = Q_0\left(1 - \frac{x_0}{h}\right)$

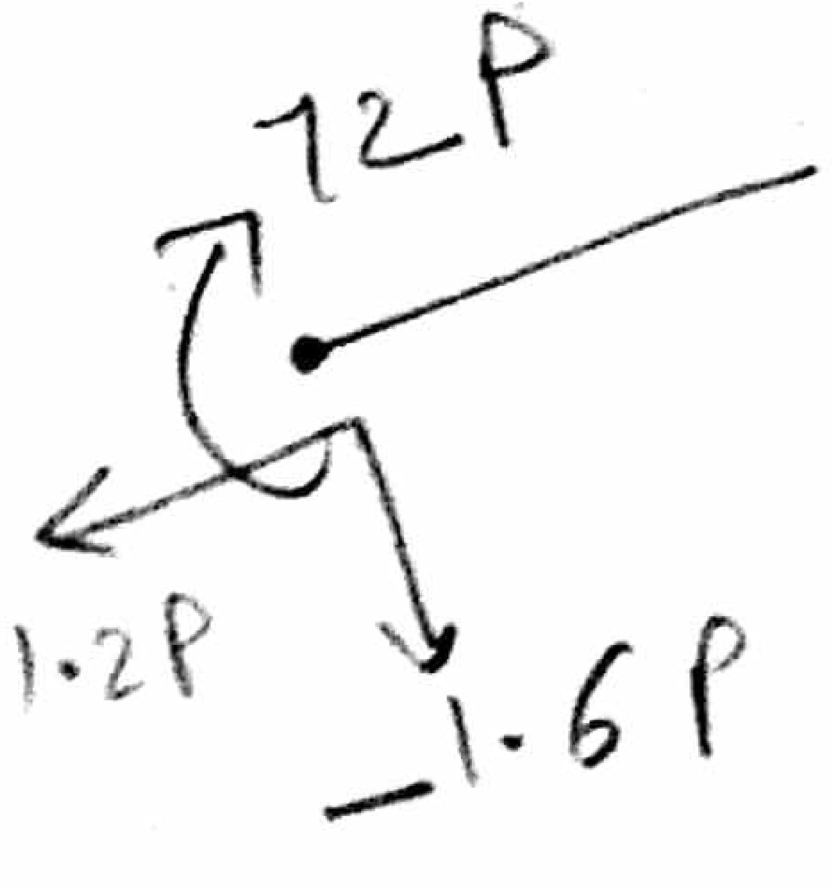
$Q_{\bullet} = Q_0\frac{x_0}{h}$

Sir's notation

If He writes $\overleftarrow{-1.2P}$ it means load 1.2P acting in left direction.



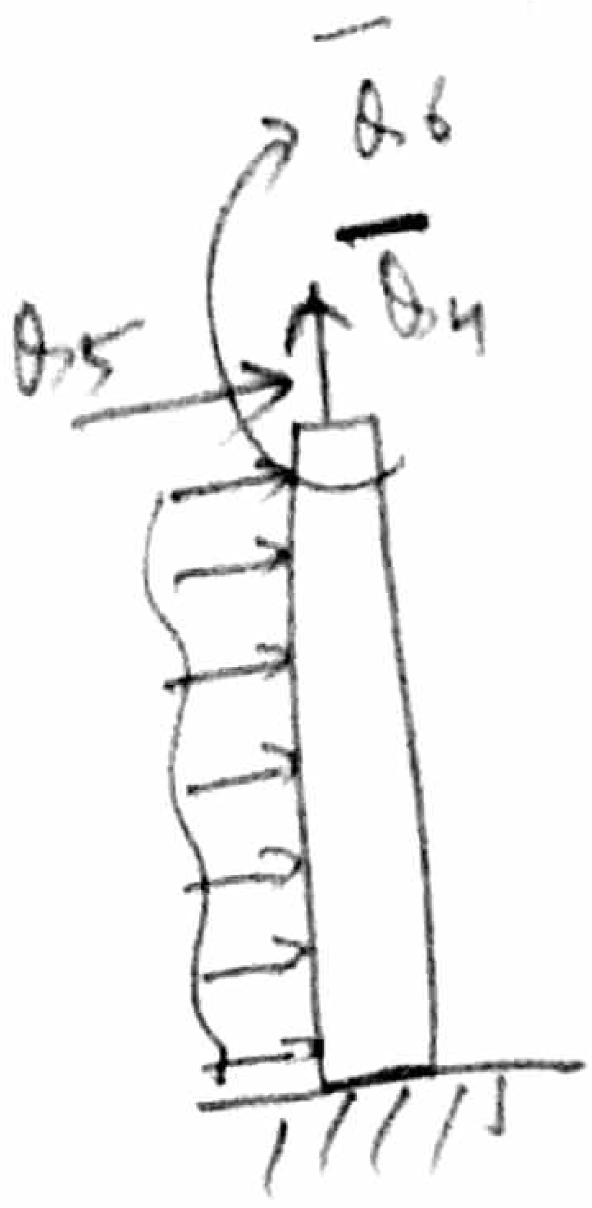


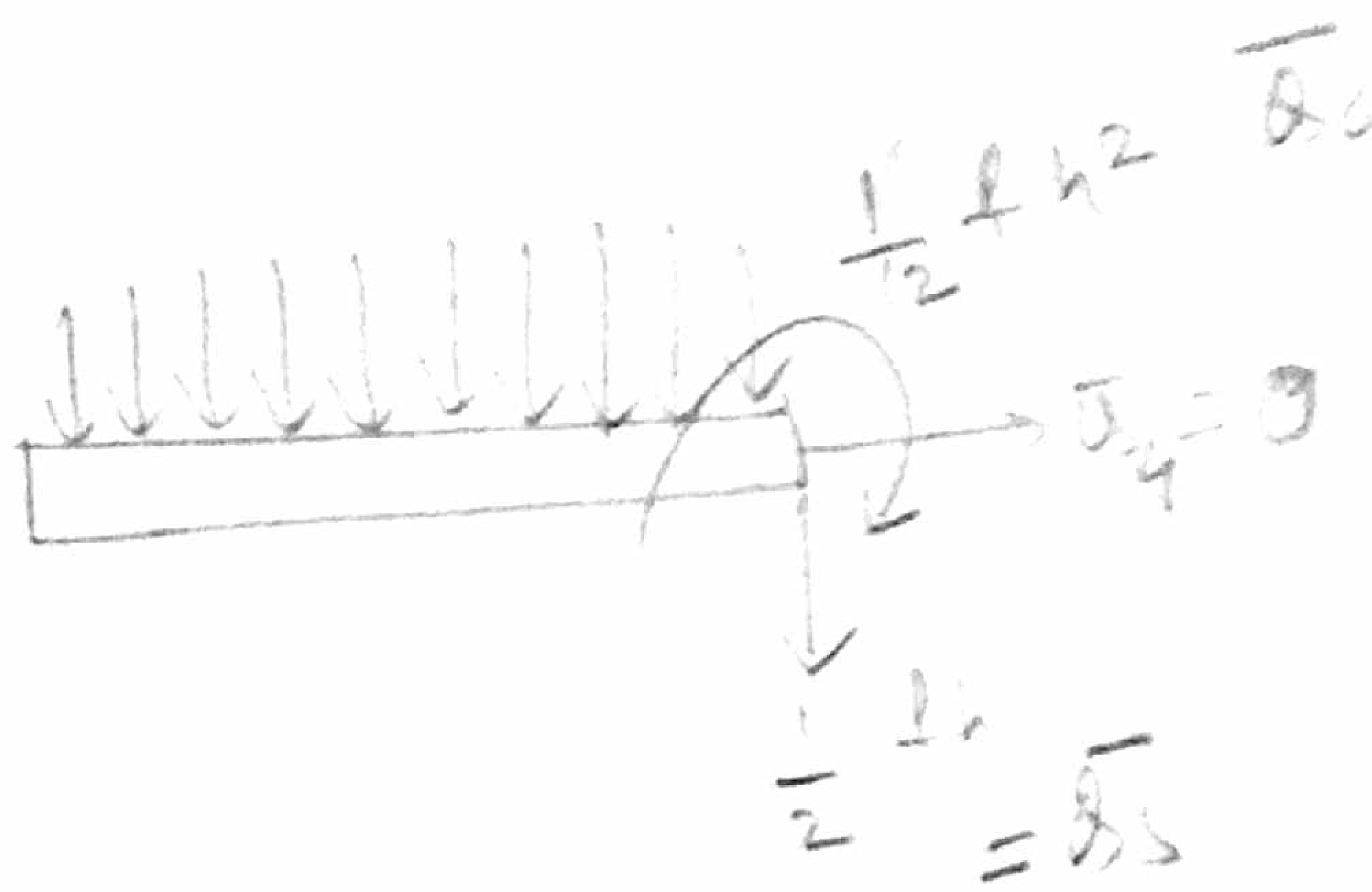


$$\phi_1 = 1 - 3\left(\frac{\bar{x}}{h}\right)^2 + 2\left(\frac{\bar{x}}{h}\right)^2 = 0.5$$

$$\phi_2 = -\bar{x}\left(1 - \frac{\bar{x}}{h}\right) = -90 * 0.5^2 = -22.5$$

$$Q = Q_0 \phi_2 = -22.5\,(-3.2P) = 72P$$





$$\bar{Q}_4 = 0$$

$$\bar{Q}_5 = \frac{1}{2}\,\frac{P}{72}\,144 = -1P$$

$$\bar{Q}_6 = -\frac{1}{12} * \frac{P}{72} * (144)^2 = -24P$$

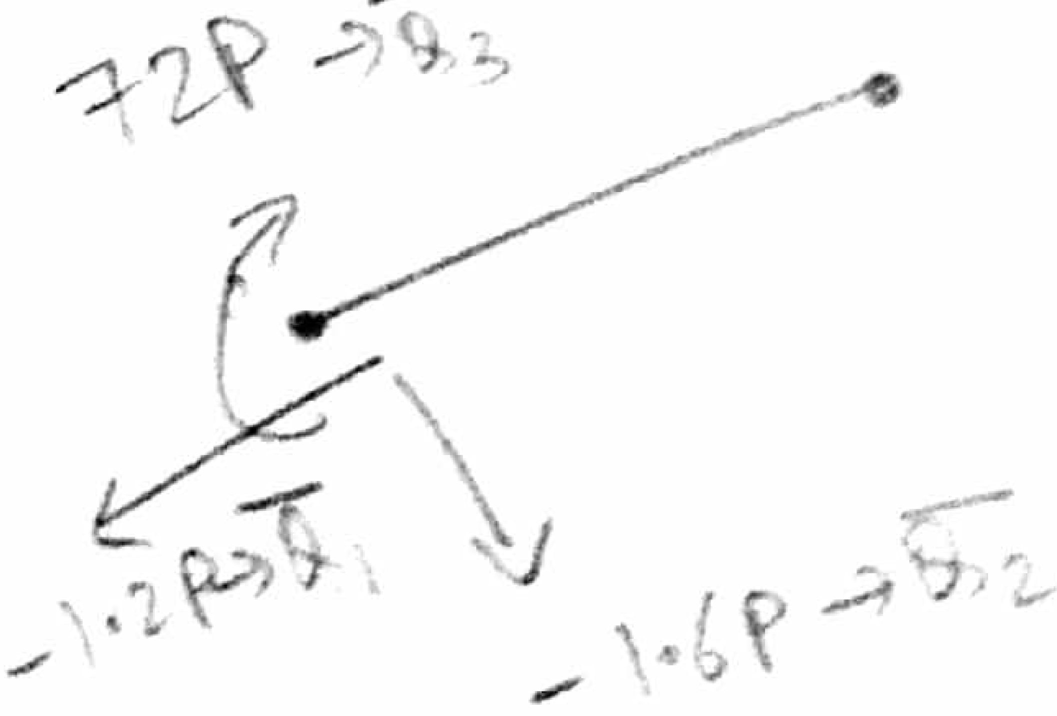
$$\begin{bmatrix} Q_4 \\ Q_5 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \bar{Q}_4 \\ \bar{Q}_5 \end{bmatrix}$$

$$Q_4 = \cos\alpha\,\bar{Q}_4 - \sin\alpha\,\bar{Q}_5 = P \quad (\cos\alpha \to 0,\ \sin\alpha \to (1),\ \bar{Q}_5 \to (-P))$$

$$Q_5 = \sin\alpha\,\bar{Q}_4 + \cos\alpha\,\bar{Q}_5 = 0 \quad (\bar{Q}_4 \to 0,\ \cos\alpha \to 0)$$

$$Q_6 = -24P$$

2nd element



$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{bmatrix}$$

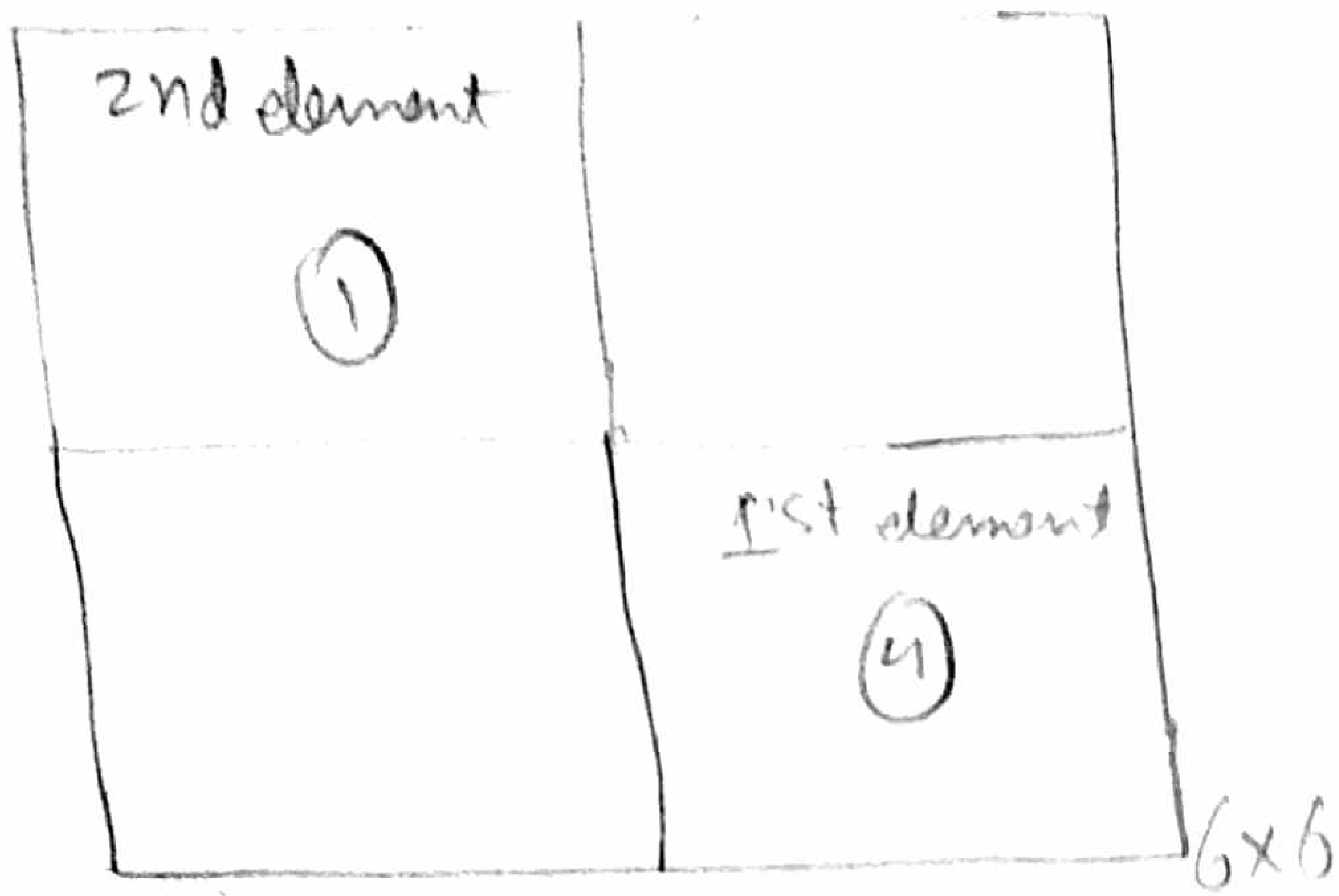
$$Q_1 = \bar{Q}_1\cos\alpha - \sin\alpha\,\bar{Q}_2$$

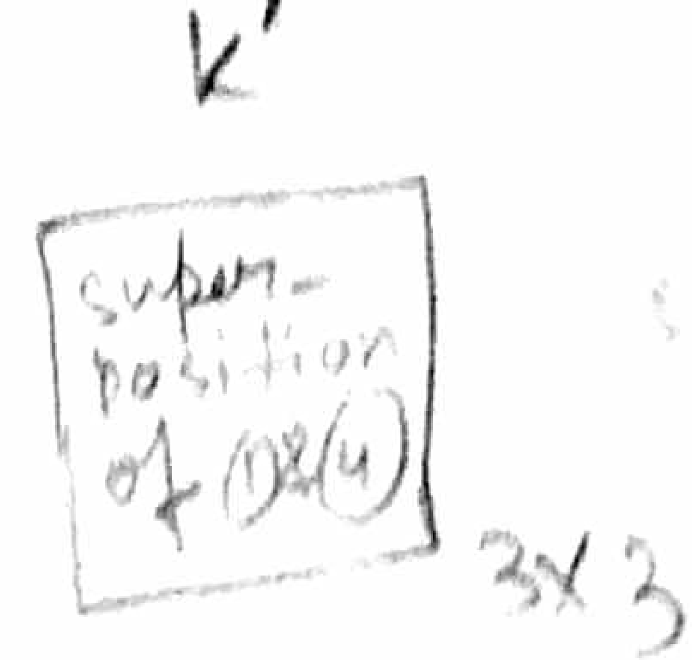
$$= 0.8(-1.2) - 0.6\,(-1.6)$$

$$Q_1 = 0$$

$$Q_2 = 0.6\,(-1.2)P + 0.8\,(-1.6)P$$

$$Q_2 = -2P$$

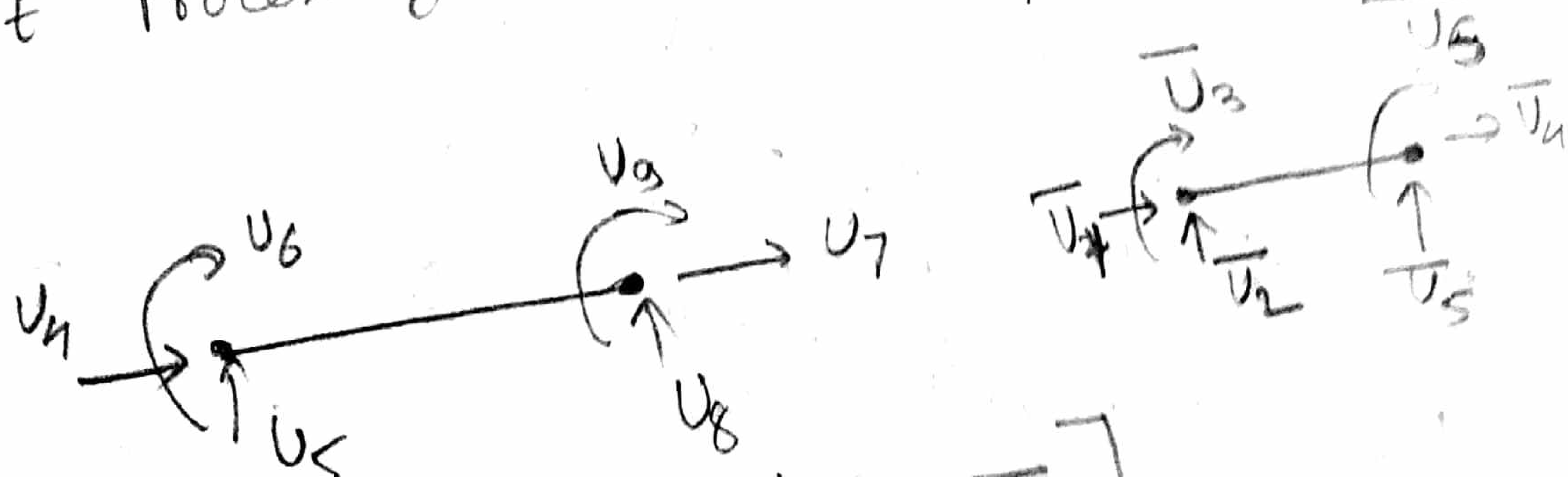




$$K' \begin{bmatrix} \\ \\ \\ \end{bmatrix} \begin{bmatrix} U_4 \\ U_5' \\ U_6 \end{bmatrix} = \begin{Bmatrix} P+0 \\ 0+(-2P) \\ -24P+72P \end{Bmatrix} + \begin{Bmatrix} 0 \\ -2P \\ 0 \end{Bmatrix} = \begin{Bmatrix} P \\ -4P \\ 48P \end{Bmatrix}$$

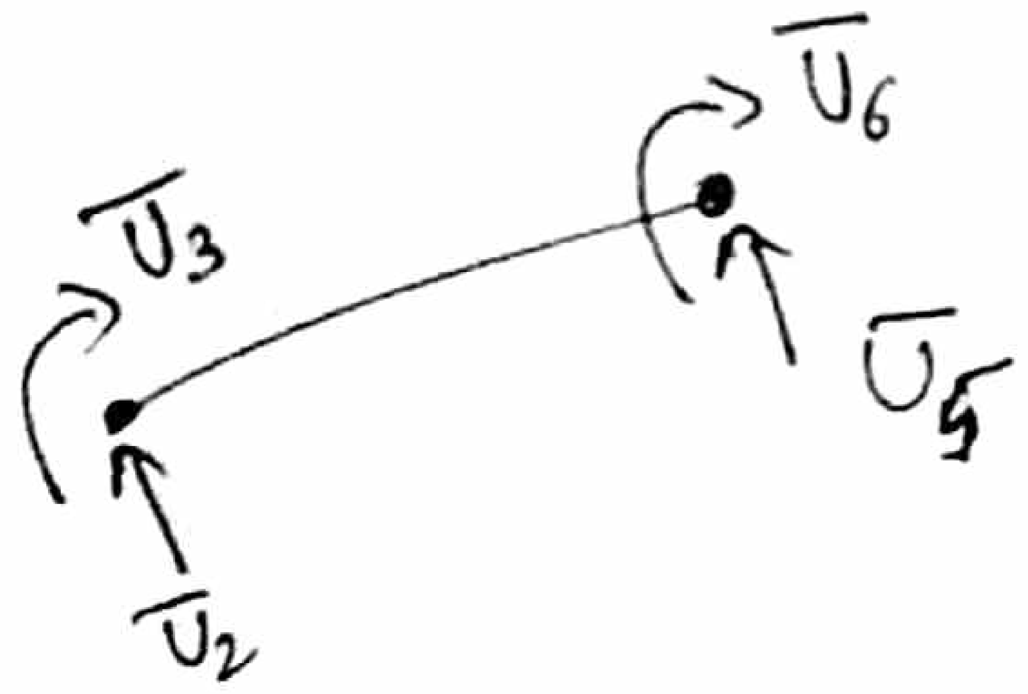
($\uparrow f$, $\uparrow Q$)

Post Processing



$$\bar{F}_4 = EA\left[\frac{d\psi_1}{dx}\bar{U}_1 + \frac{d\psi_2}{dx}\bar{U}_4\right]$$

$$f_5 = \frac{d}{dx} EI \frac{d^2 w}{dx^2} = EI \frac{d^3 w}{dx^3} = EI \sum_{j=1}^{4} \frac{d^3 \phi_j}{dx^3} \times \bar{U}_j$$



for $j = 1$, $U_j = U_2$

for $j = 2$, $U_j = U_3$

for $j = 3$, $U_j = U_5$

for $j = 4$, $U_j = U_6$

Bending moment

$$f_6 = EI \frac{d^2 w}{dx^2}$$

$$= EI \sum_{j=1}^{4} \left(\frac{d^2 \phi_j}{dx^2} \times U_j \right)$$

$$\sigma_{xx} = \frac{M_z y}{I_z}$$

* Notes:-

(1) Anything axial → Lagrangian interpolation f^n

(2) Anything transverse → Hermite interpolation f^n

$$\Pi = \int_0^h \frac{1}{2} \sigma \epsilon A \, dx - P\delta_2 - Q\delta_1$$

$$= \int_0^h \frac{1}{2} (\epsilon E) \epsilon A \, dx - P\delta_2 + Q\delta_1$$

$$= \frac{1}{2} EA \int_0^h \epsilon^2 \, dx - P\delta_2 + Q\delta_1$$

$$= \frac{1}{2} EA \int_0^h \left(\frac{du}{dx}\right)^2 - P\delta_2 + Q\delta_1$$

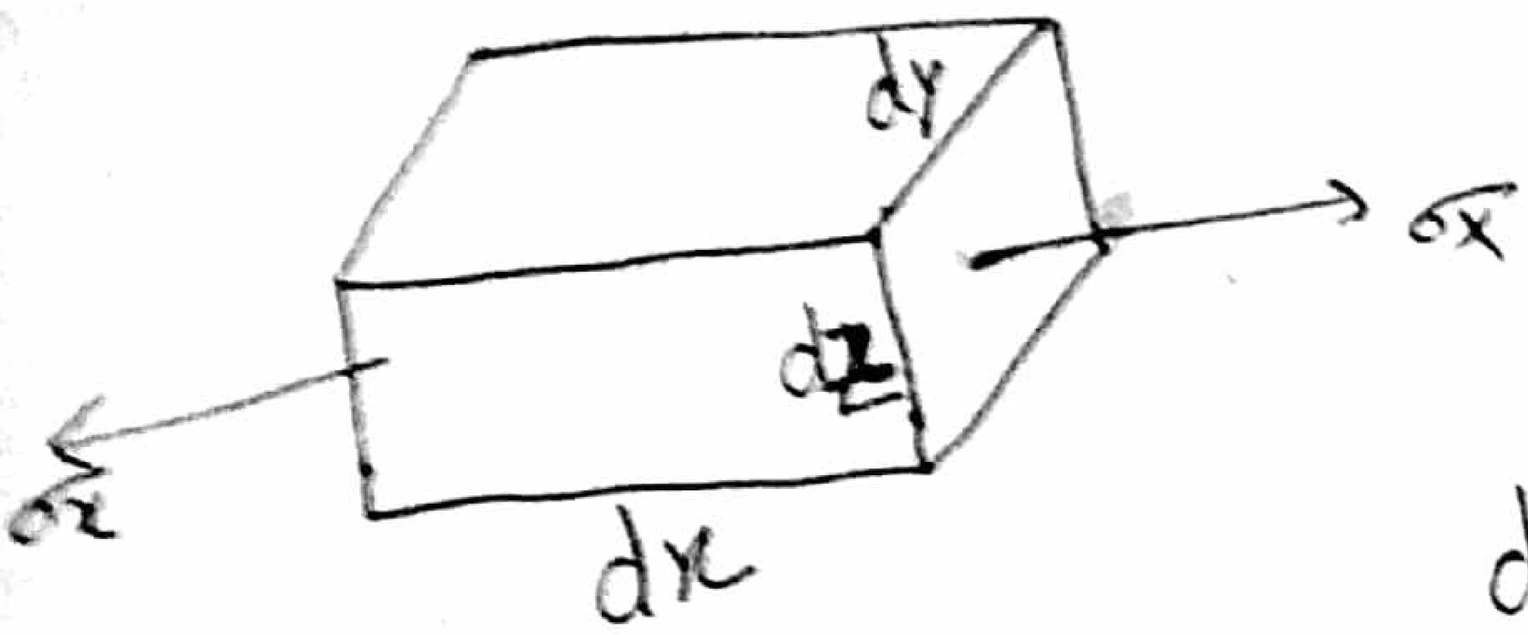
$$= \frac{1}{2} EA \int_0^h \left(\frac{d}{dx}(U_1\Psi_1 + U_2\Psi_2)\right)^2 dx - P\delta_2 + Q\delta_1 = 0$$

$$\frac{\partial \Pi}{\partial U_1} = EA \int_0^h \frac{d\Psi_1}{dx}\left(\frac{d}{dx}(\Psi_1 U_1 + \Psi_2 U_2)\right) dx + Q = 0$$

$$\frac{\partial \Pi}{\partial U_2} = EA \int_0^h \frac{d\Psi_2}{dx}\left(\frac{d}{dx}(\Psi_1 U_1 + \Psi_2 U_2)\right) dx - P = 0$$

$$\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}-R\\ P\end{Bmatrix}$$

Internal strain energy



$$dU = \sigma_x\, dy\, dz\; d\left(u + \frac{\partial u}{\partial x} dx\right) - \sigma_x\, dy\, dz\; du$$

$$dU = \sigma_x\, dy\, dz\; d\left(\frac{\partial u}{\partial x}\right) dx$$

$$dU = \sigma_x\, d\left(\frac{\partial u}{\partial x}\right) dx\, dy\, dz$$

$$dU = \sigma_x (d\epsilon_x) \underbrace{dx\, dy\, dz}_{\text{Volume}}$$

Total strain energy $= \int_0^{\epsilon_x} \sigma_x\, d\epsilon_x\, dV + \int_0^{\epsilon_y} \sigma_y\, d\epsilon_y\, dV + \int_0^{\epsilon_z} \sigma_z (d\epsilon_z)\, dV + \int_0^{\gamma_{xy}} \tau_{xy}\, d\gamma_{xy}\, dV + \int_0^{\gamma_{xz}} \tau_{xz}\, d\gamma_{xz}\, dV + \int_0^{\gamma_{yz}} \tau_{yz}\, d\gamma_{xz}\, dV = U$

U_0 = strain energy / unit volume

(A) → Modulus of resilience

(A) + (B) → Modulus of toughness

$$dU_0 = \sigma_x\, d\epsilon_x + \sigma_y\, d\epsilon_y + \sigma_z\, d\epsilon_z + \tau_{xz}\, d\gamma_{xz} + \tau_{xy}\, d\gamma_{xy} + \tau_{yz}\, d\gamma_{yz}$$

$$dU_0 = \frac{\partial U_0}{\partial \epsilon_x} d\epsilon_x + \frac{\partial U_0}{\partial \epsilon_y} d\epsilon_y + \frac{\partial U_0}{\partial \epsilon_z} d\epsilon_z + \frac{\partial U_0}{\partial \gamma_{xz}} d\gamma_{xz} + \frac{\partial U_0}{\partial \gamma_{xy}} d\gamma_{xy} + \frac{\partial U_0}{\partial \gamma_{yz}} d\gamma_{yz}$$

On comparing above two equations

$$\boxed{\sigma_x = \frac{\partial U_0}{\partial x}}$$

→ Strain energy density has the property that partial derivative of U_0 w.r.t any strain gives the corresponding stress component.

$$\left\{\frac{\partial U_0}{\partial \epsilon}\right\} = \{\sigma\} = [C]\{\epsilon\}$$

Integrating w.r.t $d\epsilon$, $U_0 = \frac{1}{2}\{\epsilon\}^T [C]\{\epsilon\}$

$$U = \frac{1}{2}\int_V \{\epsilon\}^T [C]\{\epsilon\}\, dV \rightarrow \text{strain energy}$$

External force effect

$$W = -\int_V \{u\}^T \{X\}\, dV - \int_S \{u\}^T \{P\}\, ds$$

$$\int_V (X_b u + Y_b v + Z_b w)\, dV \qquad \int_S (X_s u + Y_s v + Z_s w)\, ds$$

Total strain energy $(\pi) = \frac{1}{2}\int_V \{\epsilon\}^T [C]\{\epsilon\}\, dV - \int_S \{u\}^T \{P\}\, ds - \int_V \{u\}^T \{X\}\, dV$

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} \Rightarrow \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \rightarrow \sum u_i \psi_i \\ v \rightarrow \sum v_i \psi_i \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \psi_1}{\partial x} & 0 & \frac{\partial \psi_2}{\partial x} & 0 & \frac{\partial \psi_3}{\partial x} & 0 \\ 0 & \frac{\partial \psi_1}{\partial y} & 0 & \frac{\partial \psi_2}{\partial y} & 0 & \frac{\partial \psi_3}{\partial y} \\ \frac{\partial \psi_1}{\partial y} & \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_2}{\partial y} & \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_3}{\partial y} & \frac{\partial \psi_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$B \rightarrow$ Interpolation operator matrix

$d \rightarrow$ Displacement matrix

$$\{\epsilon\} = [B]\{d\}$$

$\{d\} \rightarrow$ specifically nodal value whose values are constant

$$\{u\} = \{\psi\}\{d\}$$

$\{u\} \rightarrow$ generalised displacement matrix

$$\Pi = \frac{1}{2}\int_V \{d\}^T [B]^T [C] [B] \{d\} \, dV - \int_V \{d\}^T \{\psi\}^T \{x\} \, dV - \int_V \{d\}^T \{\psi\}^T \{P\} \, ds$$

The first variation of Π must be zero for all equilibrium conditions.

→ Principle of stationery potential energy.

$$\delta\Pi = \{\delta d\}^T \left[\int_V [B]^T [C] [B] \, dv \, \{d\} - \int_V [\psi]^T \{x\} \, dv - \int_S [\psi]^T \{P\} \, ds \right] = 0$$

$$\left[\int_V [B]^T [C] [B] \, dv \right] \{d\} = \left[\int_V [\psi]^T \{x\} \, dv + \int_S [\psi]^T \{P\} \, ds \right] + \{Q\} + \{f\}$$

K matrix

$\{Q\}$ → This term will come, if at any node external loads are given

$$[k]\{u\} = \{f\} + \{Q\}$$

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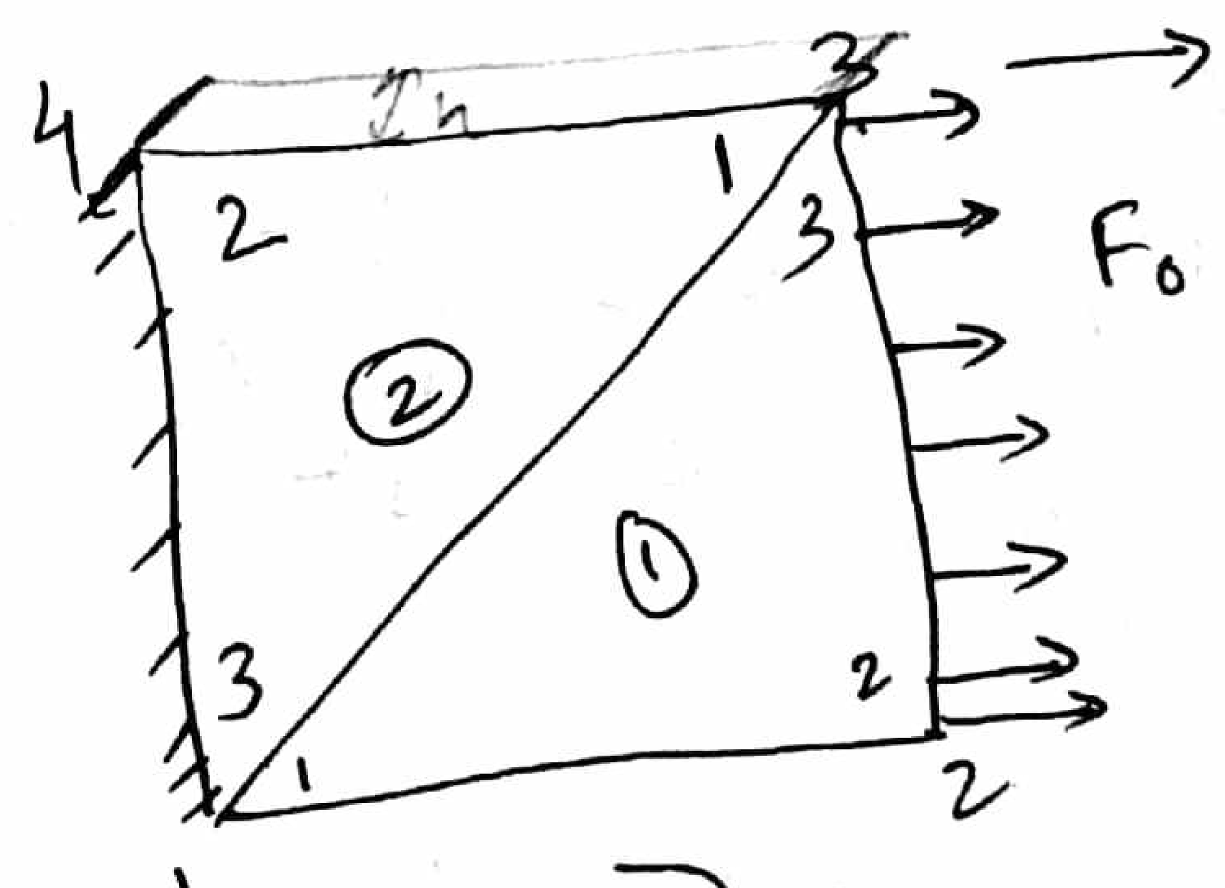
Plane stress problem

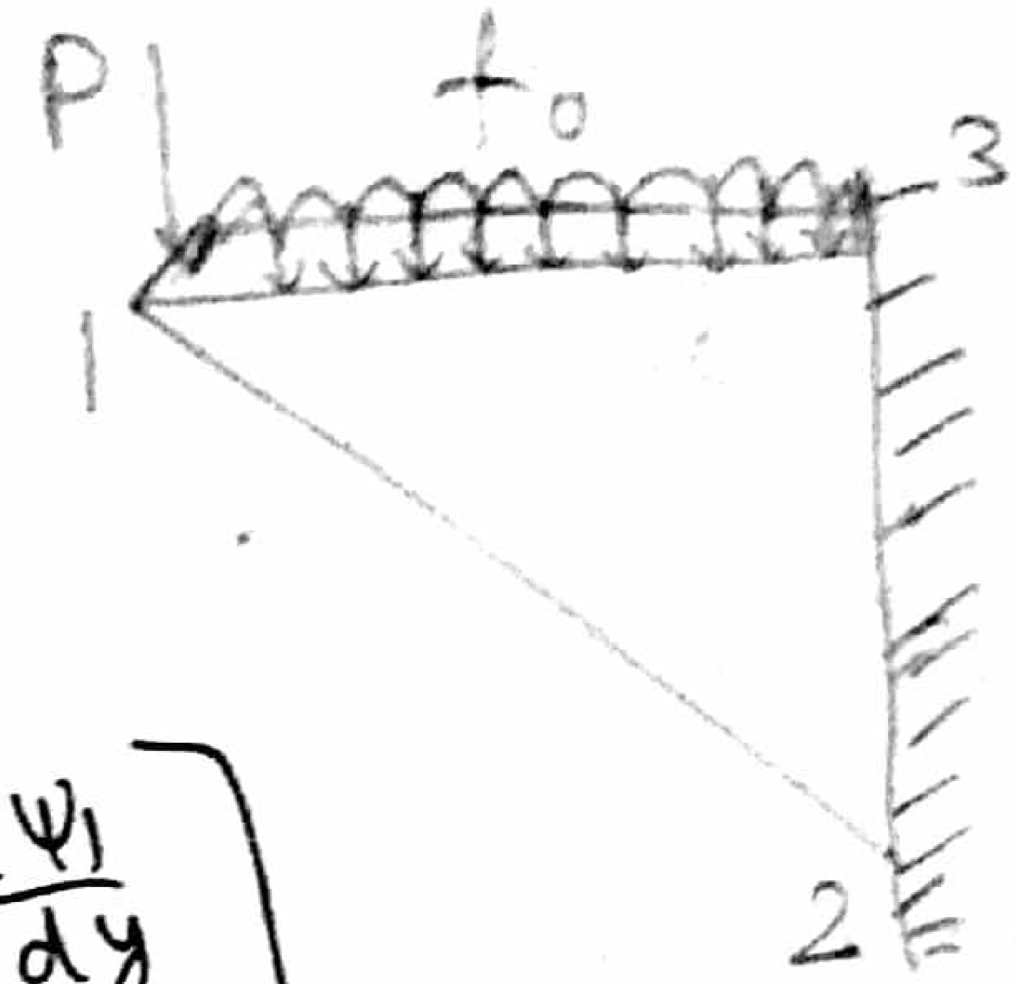


$$k_{ij} = h \int_{\Omega} B^T C B \, \underbrace{dx \, dy}_{Area}$$

↑ thickness

$$[C] = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix}, \quad [B] = \begin{bmatrix} \frac{d\phi_j}{dx} & 0 \\ 0 & \frac{d\phi_j}{dy} \\ \frac{d\phi_j}{dy} & \frac{d\phi_j}{dx} \end{bmatrix}$$

→ repeats with $j = 1, 2, 3$



$$\begin{bmatrix} \frac{d\psi_1}{dx} & 0 & \frac{d\psi_1}{dy} \\ 0 & \frac{d\psi_1}{dy} & \frac{d\psi_1}{dx} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \frac{d\psi_1}{dx} & 0 \\ 0 & \frac{d\psi_1}{dy} \\ \frac{d\psi_1}{dy} & \frac{d\psi_1}{dx} \end{bmatrix}$$

$$\begin{bmatrix} \frac{d\psi_1}{dx} & 0 & \frac{d\psi_1}{dy} \\ 0 & \frac{d\psi_1}{dy} & \frac{d\psi_1}{dx} \end{bmatrix} \begin{bmatrix} C_{11}\frac{d\psi_1}{dx} & C_{12}\frac{d\psi_1}{dy} \\ C_{21}\frac{d\psi_1}{dx} & C_{22}\frac{d\psi_1}{dy} \\ C_{66}\frac{d\psi_1}{dy} & C_{66}\frac{d\psi_1}{dx} \end{bmatrix}$$

$$C_{11}\left(\frac{d\psi_1}{dx}\right)^2 + C_{66}\left(\frac{d\psi_1}{dy}\right)^2 \qquad C_{12}\frac{d\psi_1}{dy}$$

$$h\int_A \begin{bmatrix} C_{11}\frac{d\psi_1}{dx}\cdot\frac{d\psi_1}{dx} + C_{66}\frac{d\psi_1}{dy}\frac{d\psi_1}{dy} & C_{12}\frac{d\psi_1}{dy}\frac{d\psi_1}{dx} + C_{66}\frac{d\psi_1}{dx}\frac{d\psi_1}{dy} \\ C_{21}\frac{d\psi_1}{dx}\frac{d\psi_1}{dy} + C_{66}\frac{d\psi_1}{dy}\frac{d\psi_1}{dx} & C_{22}\frac{d\psi_1}{dy}\frac{d\psi_1}{dy} + C_{66}\frac{d\psi_1}{dx}\frac{d\psi_1}{dx} \end{bmatrix} dx\,dy = K$$

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{Bmatrix} -\frac{f_0 h}{2} \\ -\frac{f_0 h}{2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -P \end{Bmatrix}$$

For triangular element

$$\psi_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$

$$\frac{\partial\psi_1}{\partial x} = \frac{\beta_i}{2A} \qquad \frac{\partial\psi_1}{\partial y} = \frac{\gamma_i}{2A}$$

$$h\int_A C_{11}\frac{d\psi_1}{dx}\frac{d\psi_1}{dx}\,dx\,dy = C_{11}\frac{\beta_1^2 h}{4A}$$

$$K = \begin{bmatrix} \frac{h C_{11} \beta_1^2}{4A} + C_{66} \frac{h \gamma_1^2}{4A} & \frac{C_{12} h \beta_1 \gamma_1}{4A} + C_{66} \frac{\beta_1 \gamma_1 h}{4A} \\ \frac{C_{21} h \beta_1 \gamma_1}{4A} + \frac{C_{66} \beta_1 \gamma_1 h}{4A} & h C_{22} \frac{\gamma_1^2}{4A} + C_{66} \frac{\beta_1^2 h}{4A} \end{bmatrix}$$

$$K = \frac{h}{4A} \begin{bmatrix} (C_{11} \beta_1^2 + C_{66} \gamma_1^2) & (C_{12} + C_{66}) \beta_1 \gamma_1 \\ (C_{12} + C_{66}) \beta_1 \gamma_1 & (C_{22} \gamma_1^2 + C_{66} \beta_1^2) \end{bmatrix}$$