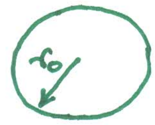
2.

When the shaft rotates alone,



shaft

$\sigma_r = 0$ at $r = r_0$, $r_0 > a$

$$\therefore \; -\frac{3+\nu}{8}\rho\omega^2 r_0^2 + A = 0 \quad [\text{for solid shaft } B = 0]$$

$$\Rightarrow A = \frac{3+\nu}{8}\rho\omega^2 r_0^2$$

$$\therefore \; \sigma_r\Big|_{shaft} = \frac{3+\nu}{8}\rho\omega^2 (r_0^2 - r^2)$$

$$\& \; \sigma_\theta\Big|_{shaft} = -\frac{1+3\nu}{8}\rho\omega^2 r^2 + \frac{3+\nu}{8}\rho\omega^2 r_0^2$$

$$\epsilon_\theta\Big|_{shaft} = \frac{u_r}{r_0} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

$$\therefore \; u_r(r_0)\Big|_{shaft} = \frac{r_0}{E}\left[-\frac{1+3\nu}{8}\rho\omega^2 r_0^2 + \frac{3+\nu}{8}\rho\omega^2 r_0^2 - \nu . 0\right]$$

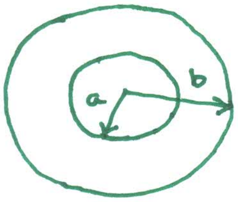
$$= \frac{\rho(1-\nu)}{4E}\omega^2 r_0^3$$

When the disc rotates alone,

$\sigma_r = 0$ at $r = a$ and b

$$\therefore \; -\frac{3+\nu}{8}\rho\omega^2 a^2 + A_1 + \frac{B_1}{a^2} = 0$$

$$-\frac{3+\nu}{8}\rho\omega^2 b^2 + A_1 + \frac{B_1}{b^2} = 0$$



disc $(r_0 > a)$

$$\Rightarrow A_1 = \frac{3+\nu}{8}\rho\omega^2(a^2 + b^2)$$

$$B_1 = -\frac{3+\nu}{8}\rho\omega^2 a^2 b^2$$

$$\therefore \; \sigma_\theta(a)\Big|_{disc} = -\frac{1+3\nu}{8}\rho\omega^2 a^2 + \frac{3+\nu}{8}\rho\omega^2(a^2+b^2) + \frac{3+\nu}{8}\rho\omega^2 b^2$$

$$= \frac{\rho\omega^2(1-\nu)}{4}a^2 + \frac{3+\nu}{4}\rho\omega^2 b^2$$

$$u_r(a)\Big|_{disc} = a\,\epsilon_\theta(a)\Big|_{disc} = \frac{a}{E}(\sigma_\theta - \nu\sigma_r)\Big|_{r=a,\,disc}$$

$$= \frac{\rho\omega^2(1-\nu)a^3}{4E} + \frac{3+\nu}{4E}\rho\omega^2 a b^2$$

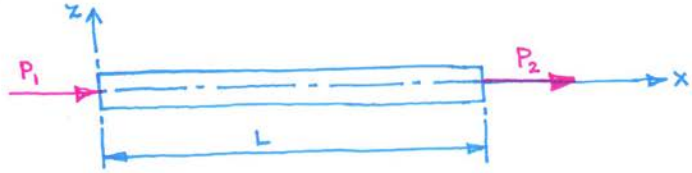
$\therefore$ The radial interference, δ is given by

$$\delta = r_0 + u_r(r_0)\Big|_{shaft} - \left(a + u_r(a)\Big|_{disc}\right)$$

$$= \frac{\rho(1-\nu)\omega^2}{4E}(r_0^3 - a^3) - \frac{3+\nu}{4E}\rho\omega^2 a b^2 + r_0 - a$$

Proved

Buckling of Column



For equilibrium: $P_2 = -P_1$

let $|P_1| = P \quad \therefore P_2 = -P$

$$\Rightarrow \int_A \sigma_x \, dA + P_1 = 0 \Rightarrow \int_A \sigma_x \, dA = -P_1 = -P$$

Displacement field: $u_0 - z\frac{\partial w_0}{\partial x}$, $w = w_0$

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 - z\frac{\partial^2 w_0}{\partial x^2} , \quad \sigma_x = E\epsilon_x$$

Internal virtual work, $\delta U = \int_\Omega \sigma_x \, \delta\epsilon_x \, d\Omega$

$$\text{or,} \quad \delta U = \int_0^L \int_A \sigma_x \left[\delta\left(\frac{\partial u_0}{\partial x}\right) + \frac{1}{2}\delta\left(\frac{\partial w_0}{\partial x}\right)^2 - z\,\delta\left(\frac{\partial^2 w_0}{\partial x^2}\right)\right] dA\, dx$$

$$= \int_0^L \int_A \sigma_x \left[\delta\left(\frac{\partial u_0}{\partial x}\right) + \frac{\partial w_0}{\partial x}\delta\left(\frac{\partial w_0}{\partial x}\right)\right] dA\, dx - \int_0^L \int_A \sigma_x \, z\, \delta\left(\frac{\partial^2 w_0}{\partial x^2}\right) dA\, dx$$

$$= -\int_0^L P\left[\delta\left(\frac{\partial u_0}{\partial x}\right) + \frac{\partial w_0}{\partial x}\delta\left(\frac{\partial w_0}{\partial x}\right)\right] dx - \int_0^L \int_A E\left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 - z\frac{\partial^2 w_0}{\partial x^2}\right] z\,\delta\left(\frac{\partial^2 w_0}{\partial x^2}\right) dA\, dx$$

$$= -\int_0^L P\left[\delta\left(\frac{\partial u_0}{\partial x}\right) + \frac{\partial w_0}{\partial x}\delta\left(\frac{\partial w_0}{\partial x}\right)\right] dx + EI\int_0^L \frac{\partial^2 w_0}{\partial x^2}\delta\left(\frac{\partial^2 w_0}{\partial x^2}\right) dx$$

$\because$ x-axis is a centroidal axis, $\int_A z\, dA = 0$, $\int_A z^2 dA = I$

$$\text{Now,} \quad -\int_0^L P\,\delta\left(\frac{\partial u_0}{\partial x}\right) dx = -\int_0^L P\frac{\partial}{\partial x}(\delta u_0)\, dx = -P\delta u_0\Big|_0^L + \int_0^L \cancelto{0}{\frac{\partial P}{\partial x}}\,\delta u_0\, dx$$

[$\because$ P is const.]

$$-\int_0^L P \frac{\partial w_0}{\partial x} \delta\left(\frac{\partial w_0}{\partial x}\right) dx = -\int_0^L P \frac{\partial w_0}{\partial x} \frac{\partial}{\partial x}(\delta w_0)\, dx$$

$$= -P \frac{\partial w_0}{\partial x} \delta w_0 \Big|_0^L + \int_0^L P \frac{\partial^2 w_0}{\partial x^2} \delta w_0\, dx$$

$$EI \int_0^L \frac{\partial^2 w_0}{\partial x^2} \delta\left(\frac{\partial^2 w_0}{\partial x^2}\right) dx = EI \int_0^L \frac{\partial^2 w_0}{\partial x^2} \frac{\partial}{\partial x}\left\{\delta\left(\frac{\partial w_0}{\partial x}\right)\right\} dx$$

$$= EI \frac{\partial^2 w_0}{\partial x^2} \delta\left(\frac{\partial w_0}{\partial x}\right)\Big|_0^L - EI \int_0^L \frac{\partial^3 w_0}{\partial x^3} \delta\left(\frac{\partial w_0}{\partial x}\right) dx$$

$$= EI \frac{\partial^2 w_0}{\partial x^2} \delta\left(\frac{\partial w_0}{\partial x}\right)\Big|_0^L - EI \frac{\partial^3 w_0}{\partial x^3} \delta w_0\Big|_0^L + EI \int_0^L \frac{\partial^4 w_0}{\partial x^4} \delta w_0\, dx$$

External virtual work: $\delta V = -P_1 \delta u_0(0) - P_2 \delta u_0(L)$
$= -P \delta u_0(0) + P \delta u_0(L)$

Virtual work principle: $\delta \Pi = \delta U + \delta V = 0$ yields

$$-P\cancel{\delta u_0}(L) + P\cancel{\delta u_0}(0) - P \frac{\partial w_0}{\partial x} \delta w_0\Big|_0^L + EI \frac{\partial^2 w_0}{\partial x^2} \delta\left(\frac{\partial w_0}{\partial x}\right)\Big|_0^L$$

$$- EI \frac{\partial^3 w_0}{\partial x^3} \delta w_0\Big|_0^L + \int_0^L \left(EI \frac{\partial^4 w_0}{\partial x^4} + P \frac{\partial^2 w_0}{\partial x^2}\right) \delta w_0\, dx - P\cancel{\delta u_0}(0) + P\cancel{\delta u_0}(L) = 0$$

$\therefore$ The governing equation is

$$EI \frac{\partial^4 w_0}{\partial x^4} + P \frac{\partial^2 w_0}{\partial x^2} = 0 \quad \text{or,} \quad \frac{\partial^4 w_0}{\partial x^4} + \lambda^2 \frac{\partial^2 w_0}{\partial x^2} = 0, \quad \lambda^2 = \frac{P}{EI}$$

Boundary conditions for fixed-free column:

$$w_0 = \frac{\partial w_0}{\partial x} = 0 \quad \text{at } x = 0$$

$$\frac{\partial^2 w_0}{\partial x^2} = 0 \quad \text{and} \quad \frac{\partial^3 w_0}{\partial x^3} + \lambda^2 \frac{\partial w_0}{\partial x} = 0 \quad \text{at } x = L$$

general solution of the governing equation:

$$w_0 = c_1 \sin \lambda x + c_2 \cos \lambda x + c_3 x + c_4$$

The boundary conditions yield the following equations:

$c_2 + c_4 = 0$ — ① $c_1 \lambda + c_3 = 0$ — ②

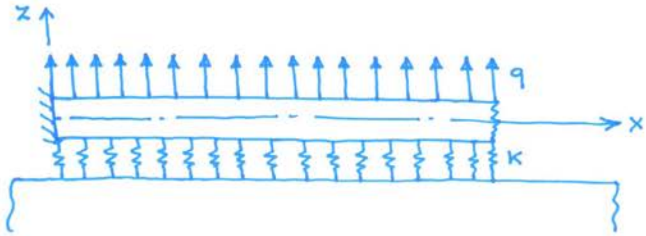
$c_1 \sin \lambda L + c_2 \cos \lambda L = 0$ — ③ and $c_3 = 0$ — ④

$\therefore$ from ②, $c_1 = 0$ from ③ $c_2 \cos \lambda L = 0$. If $c_2 = 0$, $c_4 = 0$ $\therefore$ Buckling will not occur. For buckling $c_2 \neq 0$ and $\cos \lambda L = 0$ This yields

$\lambda L = \frac{n\pi}{2}$, $n = 1, 3, 5, \cdots$ or, $\lambda^2 = \frac{n^2\pi^2}{4L^2}$ or $\frac{P}{EI} = \frac{n^2\pi^2}{4L^2}$

P will be minimum if $n = 1$. $\therefore$ the critical buckling load is $P_{cr} = \frac{\pi^2 EI}{4L^2}$

Semi-infinite beam on elastic foundation



$$u = -z\frac{\partial w_o}{\partial x}, \quad w = w_o, \quad \epsilon_x = -z\frac{\partial^2 w_o}{\partial x^2}$$

$$\delta U_1 = \int_0^\infty \int_A \sigma_x \,\delta\epsilon_x \, dA\, dx = \int_0^\infty \int_A E \frac{\partial^2 w_o}{\partial x^2} z^2 \,\delta\left(\frac{\partial^2 w_o}{\partial x^2}\right) dA\, dx$$

$$= \int_0^\infty EI \frac{\partial^2 w_o}{\partial x^2}\,\delta\left(\frac{\partial^2 w_o}{\partial x^2}\right) dx = EI\frac{\partial^2 w_o}{\partial x^2}\,\delta\left(\frac{\partial w_o}{\partial x}\right)\Big|_0^\infty - \int_0^\infty EI\frac{\partial^3 w_o}{\partial x^3}\,\delta\left(\frac{\partial w_o}{\partial x}\right) dx$$

$$= EI\frac{\partial^2 w_o}{\partial x^2}\,\delta\left(\frac{\partial w_o}{\partial x}\right)\Big|_0^\infty - EI\frac{\partial^3 w_o}{\partial x^3}\,\delta w_o\Big|_0^\infty + \int_0^\infty EI\frac{\partial^4 w_o}{\partial x^4}\,\delta w_o\, dx$$

$$\delta U_2 = \frac{1}{2}\int_0^\infty K\,\delta(w_o)^2\, dx = \int_0^\infty K w_o\,\delta w_o\, dx$$

$$\delta V = -\int_0^\infty q\,\delta w_o\, dx$$

virtual work principle: $\delta\Pi = \delta U_1 + \delta U_2 + \delta V = 0 \Rightarrow$

$$EI\frac{\partial^2 w_o}{\partial x^2}\,\delta\left(\frac{\partial w_o}{\partial x}\right)\Big|_0^\infty - EI\frac{\partial^3 w_o}{\partial x^3}\,\delta w_o\Big|_0^\infty + \int_0^\infty\left(EI\frac{\partial^4 w_o}{\partial x^4} + K w_o - q\right)\delta w_o\, dx = 0$$

$\Rightarrow$ Governing equation:

$$EI\frac{\partial^4 w_o}{\partial x^4} + K w_o = q$$

Boundary conditions: $w_o = \frac{\partial w_o}{\partial x} = 0$ at $x = 0$

$\frac{\partial^2 w_o}{\partial x^2} = \frac{\partial^3 w_o}{\partial x^3} = 0$ at $x = \infty$

General Solution: $w_o = B_1 e^{\beta x}\cos\beta x + B_2 e^{\beta x}\sin\beta x + B_3 e^{-\beta x}\cos\beta x + B_4 e^{-\beta x}\sin\beta x + \frac{q}{K}$

$B_1 = B_2 = 0$ because of $\frac{\partial^2 w_o}{\partial x^2} = \frac{\partial^3 w_o}{\partial x^3} = 0$ at $x = \infty$

$w_o = 0$ at $x = 0 \Rightarrow B_3 + \frac{q}{K} = 0 \quad \therefore B_3 = -\frac{q}{K}$

$\frac{\partial w_o}{\partial x} = 0$ at $x = 0 \Rightarrow -B_3\beta + B_4\beta = 0 \quad \therefore B_4 = B_3 = -\frac{q}{K}$

$\therefore$ The solution of deflection is

$$W_o = -\frac{q}{K}\left[e^{-\beta x}(\cos\beta x + \sin\beta x) - 1\right]$$

For maximum bending moment it must be that

$$\frac{dM}{dx} = 0 \quad \text{and} \quad \frac{d^2M}{dx^2} < 0 \quad \text{at a location.}$$

Now,  $\Rightarrow -M_1 + M = 0$ or, $M_1 = M$

Also  $\Rightarrow -M_1 - \int_A \sigma_x dA . z = 0$

$$\text{or,} \quad M_1 = -\int_A E \epsilon_x z \, dA = \int_A E z^2 \frac{\partial^2 w_o}{\partial x^2} dA = E \frac{\partial^2 w_o}{\partial x^2} \int_A z^2 dA$$

$$\therefore M = EI \frac{\partial^2 w_o}{\partial x^2}$$

Thus $\frac{dM}{dx} = 0 \Rightarrow \frac{\partial^3 w_o}{\partial x^3} = 0 \Rightarrow -\frac{4\beta^3 q}{K} e^{-\beta x} \cos\beta x = 0$

$\because x = \infty$ is not feasible $\therefore \cos\beta x = 0$

or $\beta x = \frac{n\pi}{2}$, $n = 1, 3, 5, \cdots$

Now, $\frac{d^2M}{dx^2} = EI \frac{\partial^4 w_o}{\partial x^2} = -K w_o + q = +q\left[e^{-\beta x}(\cos\beta x + \sin\beta x) - 1\right] + q$

$$= q e^{-\beta x}(\cos\beta x + \sin\beta x)$$

Obviously, $\frac{d^2M}{dx^2}$ will be < 0 first if $\beta x = \frac{3\pi}{2}$

For the next higher values of βx such that $\frac{d^2M}{dx^2} < 0$, βx will be > 4 which yields decayed result.

$\therefore M$ is maximum at $\beta x = \frac{3\pi}{2}$

$$\therefore M_{max} = EI \cdot \frac{2\beta^2 q}{K} e^{-3\pi/2}\left[\cos\left(\frac{3\pi}{2}\right) - \sin\left(\frac{3\pi}{2}\right)\right]$$

$$= \frac{2EIq\beta^2}{K} e^{-3\pi/2}$$

* Care must be taken if q is downward.